Strict monotonicity of the critical threshold

Mentor: Prof. Subhajit Goswami

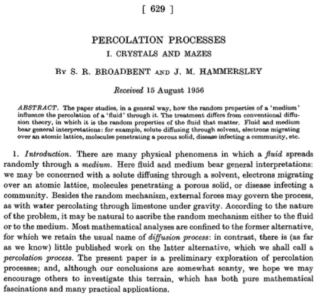
Ishaan Bhadoo

20 June 2024

Table of Contents

What is Percolation?

- Every edge of a graph G is retained with probability p and deleted with probability $1 - p$ independent of all the other edges. This random process is called **percolation.**
- This model was first introduced by Broadbent and Hammersley in 1956.



[629]

PERCOLATION PROCESSES

I. CRYSTALS AND MAZES

By S. R. BROADBENT and J. M. HAMMERSLEY

Received 15 August 1956

ABSTRACT. The paper studies, in a general way, how the random properties of a 'medium' influence the percolation of a 'fluid' through it. The treatment differs from conventional diffusion theory, in which it is the random properties of the fluid that matter. Fluid and medium bear general interpretations: for example, solute diffusing through solvent, electrons migrating over an atomic lattice, molecules penetrating a porous solid, disease infecting a community, etc.

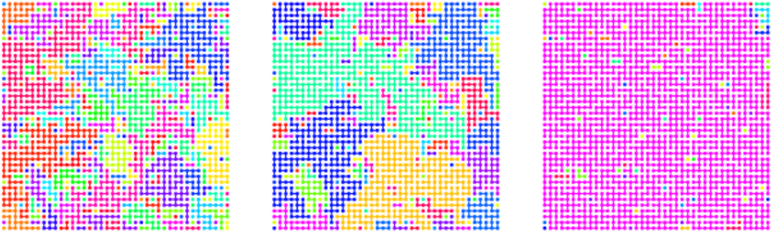
1. *Introduction.* There are many physical phenomena in which a *fluid* spreads randomly through a *medium.* Here fluid and medium bear general interpretations: we may be concerned with a solute diffusing through a solvent, electrons migrating over an atomic lattice, molecules penetrating a porous solid, or disease infecting a community. Besides the random mechanism, external forces may govern the process, as with water percolating through limestone under gravity. According to the nature of the problem, it may be natural to ascribe the random mechanism either to the fluid or to the medium. Most mathematical analyses are confined to the former alternative, for which we retain the usual name of *diffusion process*: in contrast, there is (as far as we know) little published work on the latter alternative, which we shall call a *percolation process.* The present paper is a preliminary exploration of percolation processes; and, although our conclusions are somewhat scanty, we hope we may encourage others to investigate this terrain, which has both pure mathematical fascinations and many practical applications.

(a) The Original Paper

(b) John Hammersley

An Illustration

✧ Retained edges are called **open** and deleted edges **closed**, connected components are called **clusters.**



Probability on Trees and Networks - Rusell Lyons and Yuval Peres

Figure: Percolation on a 40×40 square grid graph at levels $p = 0.4, 0.5, 0.6$. Each cluster is given a different color.

Critical threshold and the existence of phase transition

- Now we can define $\theta(p) = \mathbb{P}_p[O \leftrightarrow \infty]$.
- We want to consider, $p_c = \sup\{p : \theta(p) = 0\}$.
- An important question is the existence of phase transition for a graph G. Mathematically this boils down to showing $p_c(G) \in (0, 1)$.
- We know that $p_c(\mathbb{Z}) = 1$. For most graphs however, there is **typically** a phase transition. Usually, the bound $p_c > 0$ is easier to establish (as we will see below). The bound $p_c < 1$ is however much harder. Through a recent result of Duminil-Copin, Goswami, et.al we know that $p_c < 1$ for all transitive graphs that grow at least quadratically.

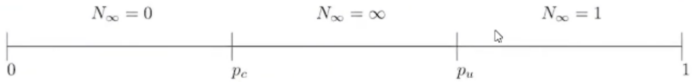


Figure: Different Phases

Transitivity of graphs and the value of p_c

Most of the classical percolation theory deals with **transitive graphs** that is graphs for which, for any two vertices $u, v \; \exists \; \gamma \in \mathrm{AUT}(G)$ s.t. $\gamma(u) = v$. The most prototypical examples of transitive graphs are Cayley graphs of finitely generated groups.

- ✧ For a d-regular tree one has $p_c(\mathbb{T}_d) = \frac{1}{d-1}$ and for the integer square lattice $p_c(\mathbb{Z}^2) = \frac{1}{2}$ (content of the next talk!).
- ✧ From the above one gets that $0 < p_c(\mathbb{Z}^d) \leq \frac{1}{2}$. However the exact value of $p_c(\mathbb{Z}^d)$ for $d \geq 3$ is not known!

For the result of the talk, we deal with graphs G (not necessarily transitive) of max degree Δ. We start by showing that for such graphs $p_c \geq \frac{1}{\Delta-1}$. In particular $p_c(G) > 0$.

Let $\Gamma_n = \{\gamma : \gamma \text{ is a SAW of length } n\}$ and let $\partial\Lambda_n = \{x : d(x, O) = n\}$.

Percolation on regular graphs

Proof.

Say, $p_c(G) < \frac{1}{\Delta-1}$ and fix a $p_c < p < \frac{1}{\Delta-1}$.
Now let,

$$X_n = \sum_{\gamma \in \Gamma_n} \mathbb{1}(\gamma \text{ is open}). \text{ Then, } \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n) \leq \mathbb{P}_p(X_n > 0) \leq \mathbb{E}_p(X_n)$$

Now,
$\mathbb{E}_p(X_n) = \sum_{\gamma \in \Gamma_n} \mathbb{P}_p(\gamma \text{ is open}) = p^n|\Gamma_n|$. However, $|\Gamma_n| \leq \Delta(\Delta-1)^{n-1}$, therefore, $\mathbb{P}_p(X_n > 0) \leq p^n\Delta(\Delta-1)^{n-1} \to 0$ as $n \to \infty$. This is a contradiction. Thus $p_c(G) \geq \frac{1}{\Delta-1}$. □

Using a similar counting argument and the second-moment method it is easy to show that for a d regular tree $p_c = \frac{1}{d-1}$.

Natural question: For what d-regular graphs does $p_c = \frac{1}{d-1}$?

Critical thresholds under coverings

Now we want to show that all transitive graphs G of degree d for which $p_c(G) = \frac{1}{d-1}$ are precisely d-regular trees.

We shall use the following result:

Theorem (S. Martineau, F. Severo '20)

Let G, H be connected graphs of bounded degree. If there exists a **nice** covering map $\pi : V(G) \to V(H)$ and $p_c(G) < 1$ then $p_c(G) < p_c(H)$.

In light of the above theorem, we will show that every **transitive** d-regular graph is covered by a graph isomorphic to the d-regular tree. More specifically we show:

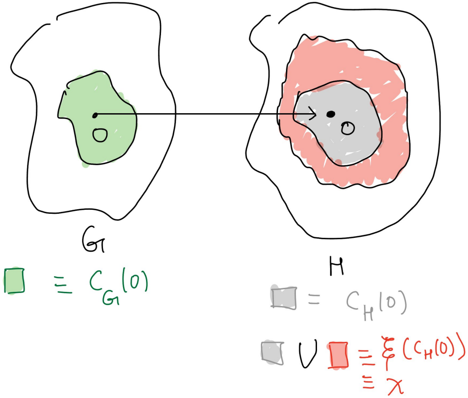
Characterization of trees

Let $\mathbb{T}_d$ be the d-regular tree and H be a transitive graph of degree d which is not isomorphic to $\mathbb{T}_d$. Then there exists a nice covering map $\pi : V(\mathbb{T}_d) \to V(H)$.

Enhancements of percolation configurations

- The main technique involved in Martineau and Severo's paper are enhancements (a local strategy), which help us compare various percolation clusters and arrive at a strict inequality between critical points (a global quantity).
- An enhancement should be thought of as a local rule with finite range, given a configuration ω the enhanced configuration of ω is $\xi(\omega) := \omega \bigcup \{\text{local changes}\}$. An example of a local rule is "If all edges coming out of u are closed, add them."

Enhancements of percolation configurations (contd.)



The idea: Construct an enhancement (say χ) of the cluster of the origin in H ($C_H(O)$) for which such a strict monotonicity holds, which is also dominated by the image of the cluster of the origin in $G(C_G(O))$.

Covering d-regular graphs

Characterization of trees

Let $\mathbb{T}_d$ be the d-regular tree and H be a transitive graph of degree d which is not isomorphic to $\mathbb{T}_d$. Then there exists a nice covering map $\pi : V(\mathbb{T}_d) \to V(H)$.

In particular $p_c(H) > \frac{1}{d-1}$, thus $p_c(H) = \frac{1}{d-1}$ only holds for trees in the class of transitive graphs of degree d.

Construction of the covering:

Fix a vertex x_0 of H. Construct a graph G with vertices as non-backtracking paths $< x_0, x_1, ...x_n >$ (i.e., $x_{i+2} \neq x_i$), where two paths are connected if one is the extension of the other by an edge.

It can be checked that for a d-regular graph H, the above constructed graph is isomorphic to $\mathbb{T}_d$.

Covering d-regular graphs (contd.)

Finally, consider the covering map $\pi : V(G) \to V(H)$, given by, $\pi(< x_0, x_1, ...x_n >) = x_n$. Then it can be shown that this is a **nice** map (this is where transitivity comes in!), and then the characterization follows.

The technical condition of checking niceness is where the transitivity of the graph H and the fact that H is not isomorphic to the tree comes in.

As we will show below, a counter-example (for $d > 2$) exists as soon as one drops transitivity.

Counter-example for the non-transitive case

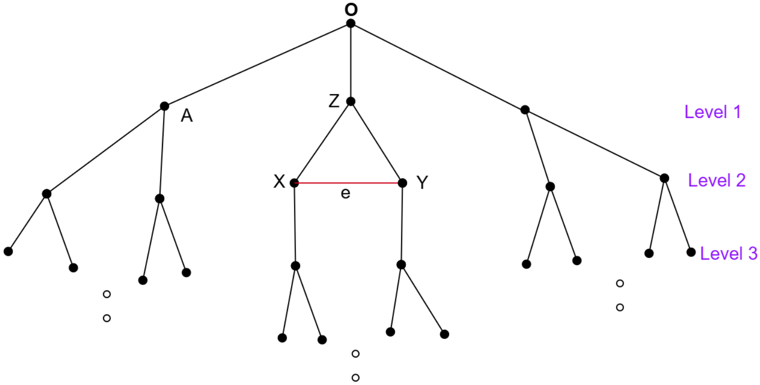


Figure: A counter for $d = 3$, a similar one can be constructed for any $d > 2$.

Conclusion and some remarks

- The graph G constructed as the covering is known as the universal cover of the graph H.
- The condition of transitivity can be relaxed to working with quasi-transitive graphs, where action by $\mathrm{AUT}(G)$ has only finitely many orbits, the theorem can be further relaxed to the case where each vertex has a cycle of length $\in (0, K)$ where K is a universal constant.
- It can be shown that free group actions by non-trivial finite groups induce a nice covering map, thus for any finite group Γ acting freely on G, and $p_c(G) < 1$ we have: $p_c(G/\Gamma) > p_c(G)$.
- Similar results as the one shown by S. Martineau and F. Severo were already known for the connective constant. A study of p_u, i.e. the uniqueness threshold was also done in their paper, in particular, the strict monotonicity also holds for p_u.

References

Paul Balister, Béla Bollobás, and Oliver Riordan.
Essential enhancements revisited.
arXiv preprint arXiv:1402.0834, 2014.

Hugo Duminil-Copin, Subhajit Goswami, Aran Raoufi, Franco Severo, and Ariel Yadin.
Existence of phase transition for percolation using the gaussian free field.
2020.

Russell Lyons and Yuval Peres.
Probability on trees and networks, volume 42.
Cambridge University Press, 2017.

Sébastien Martineau and Franco Severo.
Strict monotonicity of percolation thresholds under covering maps.
2019.

Thank You!