Percolation on Gromov Hyperbolic Graphs

Phase Transitions in the hyperbolic space

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Outline

What is Percolation?

- Every edge of a graph G is retained with probability and p and deleted with probability 1-p independent of all the other edges. This random process is called **percolation.**

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- This model was first introduced by Broadbent and Hammersley in 1956.

[629]

PERCOLATION PROCESSES

I. CRYSTALS AND MAZES

By S. R. BROADBENT AND J. M. HAMMERSLEY

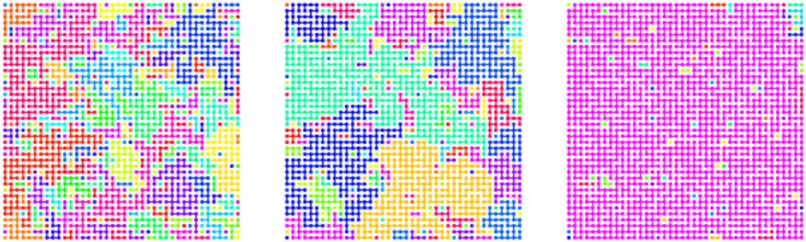
Received 15 August 1956

ABSTRACT. The paper studies, in a general way, how the random properties of a 'medium' influence the percolation of a 'fluid' through it. The treatment differs from conventional diffusion theory, in which it is the random properties of the fluid that matter. Fluid and medium bear general interpretations: for example, solute diffusing through solvent, electrons migrating over an atomic lattice, molecules penetrating a porous solid, disease infecting a community, etc.

1. *Introduction.* There are many physical phenomena in which a *fluid* spreads randomly through a *medium*. Here fluid and medium bear general interpretations: we may be concerned with a solute diffusing through a solvent, electrons migrating over an atomic lattice, molecules penetrating a porous solid, or disease infecting a community. Besides the random mechanism, external forces may govern the process, as with water percolating through limestone under gravity. According to the nature of the problem, it may be natural to ascribe the random mechanism either to the fluid or to the medium. Most mathematical analyses are confined to the former alternative, for which we retain the usual name of *diffusion process*: in contrast, there is (as far as we know) little published work on the latter alternative, which we shall call a *percolation process*. The present paper is a preliminary exploration of percolation processes; and, although our conclusions are somewhat scanty, we hope we may encourage others to investigate this terrain, which has both pure mathematical fascinations and many practical applications.

An Illustration

✧ Retained edges are called **open** and deleted edges **closed**, connected components are called **clusters.**



Probability on Trees and Networks - Ruselly Lyons and Yuval Peres

Figure: For phases before,at and after criticality

Phase Transition and Critical probabilities

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- For a d-regular tree one has that $p_c(\mathbb{T}_d) = \frac{1}{d-1}$ and for the integer square lattice $p_c(\mathbb{Z}^2) = \frac{1}{2}$.
- From the above one gets that $0 < p_c(\mathbb{Z}^d) \leq \frac{1}{2}$.However the exact value of $p_c(\mathbb{Z}^d)$ for $d \geq 3$ is not known!

Big Questions

- After what point do we get a unique infinite cluster?
- How do the two phase transitions compare to each other?

Now define $p_u = \inf\{p : \mathbb{P}_p[\exists \text{ a unique infinite cluster}] = 1\}$

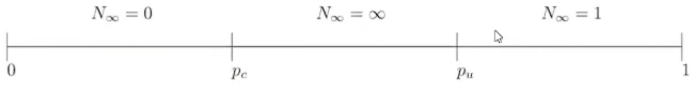

Figure: Different Phases

Now our questions translate to the following :

- What is $p_u(G)$?
- When is $p_c(G) = p_u(G)$?

Transitivity of Graphs

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- The above gives us that for infinite,connected,transitive graphs,

$$p_u = \sup\{p : \mathbb{P}_p[|N_\infty| > 1] = 1\}$$

Amenability of Graphs and Groups

Defintion

The *Isoperimetric Constant/Expansion Constant* of a graph G is given by

$$\Phi(G) = \inf\{\frac{|\partial S|}{|S|} : S \subset V, |S| < \infty\}$$

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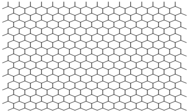
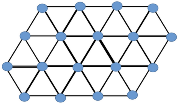
- Examples of Amenable graphs include : $\mathbb{Z}^d$, the triangular lattice , honeycomb lattice.



(a) Honeycomb Lattice



(b) Triangular Lattice

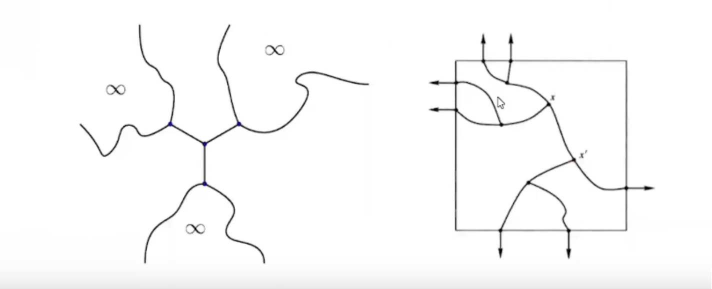
Burton and Keane's Argument

Theorem(Burton,Keane-1989)

Let G be a connected, transitive, amenable graph Then,

$$p_c(G) = p_u(G)$$

Thus for the integer lattice $p_c(\mathbb{Z}^d) = p_u(\mathbb{Z}^d)$



Tom Hutchcroft - Phase Transitions in hyperbolic spaces

A Conjecture of Benjamini and Schramm

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Let G be a infinite,connected,transitive graph then $p_c(G) = p_u(G)$ if and only if G is amenable.

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- A d regular tree for $d \geq 3$ is non-amenable and the above statement holds for it since $p_u(\mathbb{T}^d) = 1$ and $p_c(\mathbb{T}_d) = \frac{1}{d-1}$.
- (Grimmet and Newman-1990) For $b \geq 6$ we have,

$$0 < p_c(\mathbb{T}_{b+1} \square \mathbb{Z}) < p_u(\mathbb{T}_{b+1} \square \mathbb{Z}) < 1$$

- (Lalley '98, Benjamini and Schramm '01) For any planar transitive non amenable graph $p_c(G) < p_u(G)$

Hyperbolic Graphs

Definition

A graph G is said to **Gromov hyperbolic** if it satisfies the **rips thin triangle condition** that is there exists a constant δ such that for any three vertices u,v,w of G and any three geodesics [u, v], [v, w] and [w, u] between them, every point in the geodesic [u, v] is contained in the union of the δ-neighbourhoods of the geodesics [v, w] and [w, u].
Such a graph is called δ-**Hyperbolic.**

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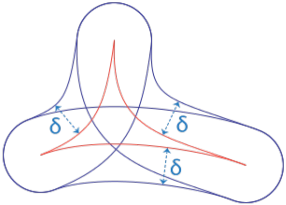


Wikipedia

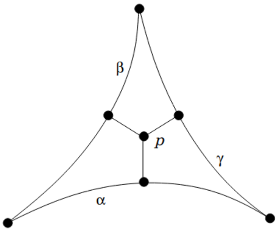
Figure: The rips thin delta condition

Alternate Definition

Definition 2

A graph G is called δ-Hyperbolic if for every geodesic triangle there exists a point $p \in V$ such that $max\{d(p,\alpha), d(p,\beta), d(p,\gamma)\} \leq \delta$.Where α, β, γ are the three geodesics.

- Both these are equivalent meaning that if G is δ-Hyperbolic in the sense of the first definition then G is δ'-Hyperbolic for some δ' in the second definition.



Bondwitch-A course on Geometric Group Theory

Non-Amenable Hyperbolic Graphs

- In 2019 Tom Hutchcroft established the Benjamini-Schramm conjecture under the additional condition of gromov hyperbolicity.
- The proof relies on important geometric properties of a hyperbolic graph and exploits these properties while using a new operator theoretic approach to the problem.
- Hutchcroft '19 also establishes the so called **Triangle condition** for such graphs which is known to imply a lot of critical exponents which describe the behaviour of percolation near criticality.

Theorem(Hutchcroft-2019)

Let G be a connected, locally finite, nonamenable, Gromov hyperbolic, quasi- transitive graph. Then $p_c(G) < p_u(G)$ and $\Delta_{p_c} < \infty$.

Thank You!