PERCOLATION OF ARBITRARY WORDS: TREES AND GRAPHS OF ISOPERIMETRIC DIMENSION GREATER THAN TWO

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Abstract. Consider site percolation on a locally finite, connected graph G satisfying $p_c(G) < 1/2$. The percolation of words problem asks whether all binary sequences can almost surely be embedded in site percolation configurations. Building on a result of Kesten and Benjamini for periodic trees, we demonstrate this for all trees extending their result. We also present the proof that all words are seen for any transitive graph of isoperimetric dimension >2. This answer the question of [BK95] and gives a new proof of [NTT23] in the case of $\mathbb{Z}^3$.

Contents

1. The problem and the results covered here

Consider Bernoulli site percolation on a graph $G = (V, E)$ with $p_c^{\text{site}}(G) < \frac{1}{2}$ and let $\omega = (\omega(v))_{v \in V}$ be a site percolation configuration. Let $\Xi = \{0, 1\}^{\mathbb{N}}$ be the set of all infinite binary sequences, referred to as the set of all words. We say that a word $\xi \in \Xi$ is *seen* from a vertex $v \in V$ if there exists a path (infinite self-avoiding path) $v = v_0 \sim v_1 \sim v_2 \sim \cdots$ such that: $\omega_i = \xi_i \quad \forall i \geq 0$.

Define, the set of words seen from a vertex, $S(v) = \{\xi \in \Xi : \xi \text{ is seen from } v\}$. We write $S_\infty(G)$ for the random set of words seen somewhere in G as $S_\infty = \bigcup_{v \in V} S(v) = \{\xi \in \Xi : \xi \text{ is seen from some } v \in V\}$.

We want to study S_∞. We operate on connected, locally finite graphs G satisfying $p_c^{\text{site}}(G) < \frac{1}{2}$ at $p = \frac{1}{2}$ (this is necessary since we want the words $(0, 0, 0, \cdots)$ and $(1, 1, 1, \cdots)$ to be seen). Let us start by stating the main problem we are interested in:

Conjecture 1 (Classification Conjecture II.(c), [Hil+14])**.** *Let G be an infinite graph, with uniformly bounded degree, and p_c^{site} denote its critical threshold for site percolation. If $p_c^{site} < p < 1 - p_c^{site}$, then all sequences can be embedded, i.e.,* $\mathbb{P}_p(S_\infty = \Xi) = 1$

The problem was introduced by Benjamini and Kesten [BK95].See Nolin–Tassion–Teixeira [NTT23] for the modern formulation and the complete solution on $\mathbb{Z}^d$, $d \geq 3$, and on sufficiently thick slabs.

We start by stating Kesten and Benjamini's theorem for periodic trees.

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Theorem 1.1 (Kesten–Benjamini [BK95])**.** *Consider a tree (T, o) with period ν, such that $p_c(T) < \frac{1}{2}$. Then*

$$S_\infty(G) = \{0,1\}^{\mathbb{N}} \qquad \textit{almost surely at } p = \frac{1}{2}.$$

Theorem 1.2 provides a natural extension of Theorem 1.1.

Theorem 1.2. *Let (T, o) be a connected locally finite tree such that $p_c^{site}(T) < \frac{1}{2}$. Then*

$$S_\infty(G) = \{0,1\}^{\mathbb{N}} \qquad \textit{almost surely at } p = \frac{1}{2}.$$

For an infinite connected graph G of maximum degree $d < \infty$, set

$$\Phi_V(r) := \min\{|\partial_V^+ K| : r \leq |K| < \infty\}, \qquad r \geq 1.$$

Following the isoperimetric-function convention of [Dis+26], define the isoperimetric dimension of G by

$$\dim_{\text{iso}}(G) := \sup\left\{D \geq 1 : \text{there is } c_D > 0 \text{ such that } \Phi_V(r) \geq c_D r^{(D-1)/D} \text{ for every } r \geq 1\right\}.$$

Thus $\dim_{\text{iso}}(G) > 2$ means precisely that, for some $D > 2$ and $c_0 > 0$,

$$\Phi_V(r) \geq c_0 r^{(D-1)/D}$$

for all sufficiently large r. We say that G satisfies *supercritical sharpness at* $1/2$ if there is $c > 0$ such that, uniformly over finite sources S and $t \geq 1$,

$$\mathbf{P}_{1/2}^S(\mathcal{C}_S \text{ is finite and } \Phi_V(\mathcal{C}_S) \geq t) \leq e^{-ct}.$$

Here the vertices of S are wired occupied, $\mathcal{C}_S$ is the union of the occupied clusters meeting S, and $\Phi_V(\mathcal{C}_S)$ is the minimum size of an external vertex boundary surrounding $\mathcal{C}_S$.

Theorem 1.3. *Let G be an infinite connected graph of bounded degree. Suppose that G has* $\dim_{\text{iso}}(G) > 2$ *and satisfies supercritical sharpness at* $1/2$*, and that Bernoulli product measure is ergodic under* $\text{Aut}(G)$*. Then*

$$S_\infty(G) = \{0,1\}^{\mathbb{N}} \qquad \textit{almost surely at } p = \frac{1}{2}.$$

Remark 1.4. Every infinite transitive graph is covered by the ergodicity assumption in Theorem 1.3. Moreover, the supercritical-sharpness theorem of Diskin–Easo–Radhakrishnan–Sudakov–Tassion [Dis+26] supplies the required finite-source estimate whenever $p_c^{\text{site}}(G) < 1/2$. Consequently every such transitive graph with $\dim_{\text{iso}}(G) > 2$ is covered by Theorem 1.3. In particular, this gives a new proof for $\mathbb{Z}^3$ of the theorem of Nolin–Tassion–Teixeira [NTT23]; their result is more quantitative and also treats sufficiently thick slabs.

Theorem 1.2 is not a corollary of Theorem 1.3. An arbitrary tree can have long bottlenecks and a poor ordinary isoperimetric profile even when its branching number is larger than 2. The tree proof therefore propagates a *flow-weighted mass*. By contrast, the general-graph proof propagates the ordinary cardinality of an intrinsic percolation ball.

Idea of the proof. Finite binary words form a binary tree. In both arguments we attach a notion of mass to each realized prefix and show that the probability that this mass fails to pass to either child is very small. There are only 2^{n+1} child extensions at level n, so a union bound shows that, with positive probability, every finite prefix is realized simultaneously. Kőnig's lemma then gives an infinite realizing path for every infinite word. For trees the mass comes from a branching-number flow; for general graphs it is the size of an intrinsic percolation ball.

Organization of the note. Section 2 proves the result for general trees. Section 3 proves the result for general graphs by combining an intrinsic-distance exploration with the finite-cluster tail supplied by supercritical sharpness. We conclude with two conjectures suggested by this argument.

Acknowledgment and statement on A.I. use. The authors thank Vincent Tassion for suggesting the problem. The proof for general trees was found by the authors. After the proof of sharpness [Dis+26], the first author prompted GPT 5.6 Pro to solve the general case using ideas from supercritical sharpness and the proof for trees. The proof for general graphs of isoperimetric dimension greater than 2 was then found autonomously by GPT 5.6 Pro with minimal discussion. We refined the argument and presentation and take full responsibility for the arguments.

2. The tree proof

In this section we prove Theorem 1.2. The proof follows two steps analogous to the proof of Theorem 1.1. Firstly we show the finiteness of a certain sum and prove that it implies our result, next we show that the sum is indeed finite. The key difference is that we use different events to do the above computation which capture more than just the growth rate of the tree.

Before we dive into the proof, we study site percolation on trees. Define the quantity,

$$\widetilde{br}T = \sup\left\{\lambda : \exists \text{ non-zero flow } \theta \text{ on the vertex set such that } 0 \leq \theta(v) \leq \lambda^{-|v|}\right\}$$

Where flows on vertices is defined in a similar way as for edges, i.e. we have a root vertex o and every vertex other than o satisfies flow conservation. Here $|v|$ is the distance of v from o.

By max flow min cut theorem (which holds for countable locally finite graphs), we have that

$$\widetilde{br}T = \sup\left\{\lambda : \inf_{\Pi}\sum_{v\in\Pi}\lambda^{-|v|} > 0\right\}$$

where the infimum is over vertex cutsets Π, i.e. Π is a finite set of vertices such that o is not connected to infinity in $T \setminus \Pi$.

Lemma 2.1. *For site percolation on T,*

$$\mathbb{P}_p(o \leftrightarrow \infty) \leq \inf\left\{\sum_{v\in\Pi} p^{|v|} : \Pi \textit{ is a (vertex) cutset separating } o \textit{ and } \infty\right\}$$

Proof. Since, $\{o \leftrightarrow \infty\} \subseteq \bigcup_{v\in\Pi}\{o \leftrightarrow v\}$ for all (vertex) cutsets Π □

We start by showing $p_c^{\text{site}} \geq \frac{1}{\widetilde{br}T}$. Consider $p > p_c$ and $\lambda = \frac{1}{p}$, then by Lemma 2.1, for all (vertex) cutsets Π

$$\sum_{v\in\Pi}\lambda^{-|v|} = \sum_{v\in\Pi}p^{|v|} > 0$$

So, $\widetilde{br}T \geq \frac{1}{p}$ for all $p > p_c$, which implies $p_c^{\text{site}} \geq \frac{1}{\widetilde{br}T}$.

For bond percolation on the tree T, a theorem of Lyons [Lyo90] states that $p_c^{\text{bond}} = \frac{1}{brT}$. Here brT is the branching number of the tree. By the same argument, a similar statement is true in our case. More specifically, we have:

Theorem 2.2. *Consider site percolation a rooted, locally finite, connected tree T. Then*

$$p_c^{site}(T) = \frac{1}{\widetilde{br}T}.$$

We want to show the following theorem,

Theorem 2.3. *Let T be a connected locally finite tree such that $p_c(T) < \frac{1}{2}$. Then $\mathbb{P}(S_\infty = \Xi) = 1$.*

First, we start by setting up some notation: Let T_n denote the set of vertices at distance n. For a word η, let $\eta^{(n)} = (\eta_1, \cdots, \eta_n)$.

For two vertices $v, w \in T$ and an infinite word $\eta \in \Xi$, we denote by $v \xleftrightarrow{\eta^{(n)}} w$ the event that there exists a self-avoiding path $v = v_0 \sim v_1 \sim \cdots \sim v_{n-1} = w$ from v to w along which $\eta^{(n)}$ is seen, i.e., such that $\omega_{v_i} = \eta_i$ for all $i \in \{0, \dots, n-1\}$. We simply write $\{v \leftrightarrow w\} := \{v \xleftrightarrow{\mathbb{1}^{(n)}} w\}$ for the monochromatic word. For trees, since there is only one path between vertices, for clarity we write (only for trees) $v \overset{\eta}{\leftrightarrow} w$. We also write $x \geq S$ (for a vertex x and set of vertices S) if x is a descendant of S. We are now ready to prove Theorem 2.3.

Proof. Suppose $\widetilde{br}T = 2 + 3\varepsilon$ for some $\varepsilon > 0$. Then by definition of the branching number there exists a non-zero flow θ such that $0 \leq \theta(v) \leq (2 + 2\varepsilon)^{-n}$. Fix this θ and let its strength be $||\theta|| = \delta > 0$.

Define random variables,

$$X_n(\eta) = \sum_{v \in T_n} (2 + \varepsilon)^n \theta(v) \mathbb{1}(o \overset{\eta}{\leftrightarrow} v)$$

Also define the events,

$$E_n(\eta) = \{X_n(\eta) \geq \delta\}$$

It is clear that if $E_n(\eta)$ occurs the word $\eta^{(n)}$ must be seen from o. For the word $\mathbb{1} = (1, 1, \cdots)$, we set $X_n := X_n(\mathbb{1})$ and $E_n := E_n(\mathbb{1})$.

Lemma 2.4. *If $\sum_{n \geq 0} 2^n \, \mathbb{P}_{\frac{1}{2}}(E_n \setminus E_{n+1}) < \infty$ then all words are seen.*

Proof. Firstly note that due to symmetry $X_n(\eta) \overset{(d)}{=} X_n(\mathbb{1})$. Thus,

$$\mathbb{P}_{\frac{1}{2}}(E_n(\eta) \setminus E_{n+1}(\eta)) = \mathbb{P}_{\frac{1}{2}}(E_n \setminus E_{n+1}) \; \forall \eta$$

Now suppose that the sum is finite. Consider the events,

$$A_n = \left\{ \text{there exists } \eta^{(n+1)} \text{ such that } E_n(\eta) \setminus E_{n+1}(\eta) \text{ holds} \right\}$$

We have, $\mathbb{P}(A_n) = \sum_{\eta^{(n+1)}} \mathbb{P}_{\frac{1}{2}}(E_n(\eta) \setminus E_{n+1}(\eta)) = 2^{n+1} \mathbb{P}_{\frac{1}{2}}(E_n \setminus E_{n+1})$, the summability condition this implies that $\sum_{n \geq 0} \mathbb{P}(A_n) < \infty$.

Thus by Borel-Cantelli we get that $\mathbb{P}(A_n \text{ i.o.}) = 0$. So there exists $N = N(\omega)$ such that A_n does not happen a.s. for $n \geq N$ i.e., for all $n \geq N$ and for all $\eta^{(n+1)}$ we have that, $E_n(\eta) \setminus E_{n+1}(\eta)$ does not occur.

Now we show that for all $n \geq 1$, there is a $\eta^{(n)}$ such that $E_n(\eta)$ occurs. Consider

$$\sum_{\eta^{(n)}} X_n(\eta) = \sum_{\eta^{(n)}} \sum_{v \in T_n} (2 + \varepsilon)^n \theta(v) \mathbb{1}(o \overset{\eta}{\leftrightarrow} v) = \sum_{v \in T_n} \sum_{\eta^{(n)}} (2 + \varepsilon)^n \theta(v) \mathbb{1}(o \overset{\eta}{\leftrightarrow} v)$$

But $\sum_{\eta^{(n)}} \mathbb{1}(v \overset{\eta}{\leftrightarrow} w) = 1$. So,

$$\sum_{\eta^{(n)}} X_n(\eta) = \sum_{v \in T_n} (2 + \varepsilon)^n \theta(v) = \delta (2 + \varepsilon)^n$$

If $X_n(\eta) < \delta$ for all $\eta^{(n)}$, then the LHS $< \delta 2^n$, which is a contradiction. Thus there must exists a $\eta^{(n)}$ such that $E_n(\eta)$ occurs. Let $n > N$, then by the above observation $E_{n+m}(\eta^{(n)}, \xi^{(m)})$ must also occur for any ξ, thus all words are seen from the generation n.

$\square$

Now we show that the sum under consideration is indeed finite, in the proof we use McDiarmid's inequality. We start by stating it:

Theorem 2.5 (McDiarmid's inequality [McD89]). *Let $X_1, \dots, X_n$ be independent random variables where X_i is $\mathcal{X}_i$-valued for all i, and let $X = (X_1, \dots, X_n)$. Assume $f : \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \to \mathbb{R}$ is a measurable function such that $\|D_i f\|_\infty < +\infty$ for all i. Then for all $\beta > 0$,*

$$\mathbb{P}[f(X) - \mathbb{E}[f(X)] \le -\beta] \le \exp\left(-\frac{2\beta^2}{\sum_{i\le n} \|D_i f\|_\infty^2}\right).$$

Here, for $x = (x_1, x_2, \cdots x_n)$ the definition of $D_i f(x)$ is:

$$D_i f(x) := \sup_{y\in\mathcal{X}_i} f(x_1, x_2, \dots, y, x_{i+1}, \dots, x_n) - \sup_{y'\in\mathcal{X}_i} f(x_1, x_2, \dots, y', x_{i+1}, \dots, x_n).$$

Lemma 2.6. *There exists $\varepsilon' > 0$ and $C > 0$ such that $\mathbb{P}(E_{n+1}^{\mathsf{c}}|E_n) \le \exp(-C(1+\varepsilon')^n)$*

Lemma 2.6 implies the summability condition in Lemma 2.4, so it remains to prove Lemma 2.6.

Proof of Lemma 2.6. Fix a configuration on $\bigcup\limits_{i\le n} T_i$ such that E_n occurs. We show the above bound conditioned on this configuration. More formally, let $\tilde{\mathbb{P}}$ be the conditional measure given $E_n, U = \{u_1, \cdots u_k\}$ where U is the set of vertices in $T_n \cap \mathcal{C}$[1] such that $\sum\limits_{1\le i\le k} (2+\varepsilon)^n \theta(u_i) \ge \delta$.

Clearly, $X_{n+1} = \sum\limits_{\substack{|x|=n+1\\ x\ge U}} (2+\varepsilon)^{n+1}\theta(x)\mathbb{1}(U \leftrightarrow x)$ and thus $\tilde{\mathbb{E}}(X_{n+1}) = \frac{(2+\varepsilon)^{n+1}}{2} \sum\limits_i \theta(u_i)$.

Now, for each descendant of x of U in T_{n+1} consider the random variable $(2+\varepsilon)^{n+1}\theta(x)\mathbb{1}(U \leftrightarrow x) \le (2+\varepsilon)^{n+1}\theta(x)$. Then by Theorem 2.5, using f as the sum of these random variables, we get that

$$\mathbb{P}[X_{n+1} - \mathbb{E}[X_{n+1}] \le -\beta] \le \exp\left(-\frac{2\beta^2}{\sum_{i\le n} \|D_i f\|_\infty^2}\right).$$

Now,

$$\sum_i \|D_i f\|_\infty^2 \le (2(2+\varepsilon)^{n+1})^2 \sum_{\substack{|x|=n+1\\ x\ge U}} (2+\varepsilon)^{n+1}\theta(x)^2 \le \frac{4(2+\varepsilon)^{2n+2}}{(2+2\varepsilon)^n} \sum_{i\le k} \theta(u_i)$$

$$= \frac{4\,(2+\varepsilon)^{n+1}}{(2+2\varepsilon)^n} \cdot 2\,\tilde{\mathbb{E}}(X_{n+1})$$

$$= \frac{8\,(2+\varepsilon)^{n+1}}{(2+2\varepsilon)^n}\,\tilde{\mathbb{E}}(X_{n+1})$$

For $\beta = \frac{\varepsilon}{2+\varepsilon}\tilde{\mathbb{E}}(X_{n+1})$, by Theorem 2.5,

[1] cluster of o

$$\tilde{\mathbb{P}}\left(X_{n+1} \leq \frac{2}{2+\varepsilon}\tilde{\mathbb{E}}(X_{n+1})\right) \leq \exp\left(-\frac{\left(\frac{\varepsilon}{2+\varepsilon}\right)^2 \tilde{\mathbb{E}}(X_{n+1})\,(2+2\varepsilon)^n}{4\,(2+\varepsilon)^{n+1}}\right).$$

Since $\tilde{\mathbb{E}}(X_{n+1}) \geq \frac{\delta\,(2+\varepsilon)}{2}$,

$$\begin{aligned}\tilde{\mathbb{P}}\left(X_{n+1} \leq \frac{2}{2+\varepsilon}\tilde{\mathbb{E}}(X_{n+1})\right) &\leq \exp\left(-\frac{1}{8}\left(\frac{\varepsilon}{2+\varepsilon}\right)^2 \delta\left(1+\frac{\varepsilon}{2+\varepsilon}\right)^n\right)\\ &= \exp\left(-C\left(1+\frac{\varepsilon}{2+\varepsilon}\right)^n\right).\end{aligned}$$

Note that,

$$\left\{X_{n+1} \geq \frac{2}{2+\varepsilon}\tilde{\mathbb{E}}(X_{n+1})\right\} \implies E_{n+1}$$

So,

$$\tilde{\mathbb{P}}(E_{n+1} \text{ does not occur}) \leq \tilde{\mathbb{P}}\left(X_{n+1} \leq \frac{2}{2+\varepsilon}\tilde{\mathbb{E}}(X_{n+1})\right) \leq \exp\left(-C\left(1+\varepsilon'\right)^n\right)$$

Where $\varepsilon' = \frac{\varepsilon}{2+\varepsilon}$ and we are done. □

□

3. Proof for general graphs

In this section we prove Theorem 1.3 for general graphs of isoperimetric dimension greater than 2. A tree has a canonical exploration structure because there is a unique path from the root to each vertex. On a general graph there may be many paths to the same vertex, so we instead explore breadth first in intrinsic distance and choose a parent (in the previous iteration) when a vertex is first discovered.

For each fixed finite word, every newly inspected vertex is explored only once. At density $1/2$ these are fresh fair Bernoulli tests, and therefore the resulting exploration has the same marginal law as the intrinsic-ball exploration of ordinary Bernoulli site percolation at parameter $1/2$. Supercritical sharpness result of [Dis+26] is then used to control the probability that such an intrinsic ball grows too slowly (despite the previous iteration being 'big'). Explorations for different words may be dependent, but the proof uses the marginal estimate for one word at a time and then takes a union bound.

The exploration procedure. Fix a finite connected set $S \subset V$. By symmetry it is enough to consider words beginning with 0, so condition throughout on every vertex of S having label 0. The labels outside S remain independent fair bits. For a finite string $\eta = (\eta_1, \ldots, \eta_n)$, begin with S and explore in intrinsic-distance order. At stage j, inspect every previously unseen neighbor of the vertices accepted at stage $j-1$, accepting it precisely when its label is η_j. Each vertex is inspected only once. Let $A_j(\eta)$ be the set accepted through stage j, including S. Choosing a parent at the first inspection gives every vertex in the jth layer a self-avoiding path from S carrying $(0, \eta_1, \ldots, \eta_j)$.

For a fixed η, every inspected vertex is accepted by a fresh Bernoulli$(1/2)$ test. Consequently $(A_j(\eta))_{0\leq j\leq n}$ has the same law as the first n intrinsic balls of ordinary Bernoulli site percolation at parameter $1/2$, with S wired occupied. This is an elementary marginal identity for the intrinsic-distance exploration.

Proof of Theorem 1.3. Let d be the maximum degree of G, and let S be a connected set of size M, where M will be chosen large. Work, as above, under the conditioning that every vertex of S has label 0. In ordinary 1/2-percolation with S wired occupied, write $\mathcal{C}_S$ for the cluster of S and

$$B_k = B_k^{\text{int}}(S) := \{x \in \mathcal{C}_S : d_{\mathcal{C}_S}(x, S) \le k\}, \qquad k \ge 0.$$

Thus B_k is simply the intrinsic ball discovered after k steps of the breadth-first exploration. The sharpness assumption gives

$$\mathbf{P}^S_{1/2}(r \le |\mathcal{C}_S| < \infty) \le e^{-c\Phi_V(r)}. \tag{3.1}$$

Indeed, a finite cluster of size at least r has surrounding boundary at least $\Phi_V(r)$. For transitive graphs with $p_c^{\text{site}}(G) < 1/2$, this is the finite-source consequence of Diskin–Easo–Radhakrishnan–Sudakov–Tassion [Dis+26].

Choose $D > 2$ and $c_0 > 0$ so that $\Phi_V(r) \ge c_0 r^{(D-1)/D}$ for all sufficiently large r. For a small constant $a > 0$, define

$$h(r) = \max\left\{1, \left\lfloor \frac{a\Phi_V(r)}{\log(edr/\Phi_V(r))} \right\rfloor\right\}, \qquad m_0 = M, \qquad m_{n+1} = \inf_{r \ge m_n} (r + h(r)).$$

We claim that an intrinsic ball is very unlikely to gain fewer than $h(r)$ vertices. On $\{|B_n| = r\}$, close the next layer $B_{n+1} \setminus B_n$. The cluster of S is then exactly B_n. If that layer has at most $h(r)$ vertices, there are at most

$$\sum_{j \le h(r)} \binom{dr}{j} \le (dr + 1) \exp\left\{h(r) \log \frac{edr}{h(r)}\right\}$$

possible preimages of the resulting configuration. At density 1/2 the closing operation does not change their weights. Taking a small and using the cluster tail (3.1) therefore gives

$$\mathbf{P}^S_{1/2}(|B_n| = r,\ |B_{n+1}| < r + h(r)) \le C(dr + 1)e^{-c\Phi_V(r)}. \tag{3.2}$$

Summing (3.2) over $r \ge m_n$ yields

$$\mathbf{P}^S_{1/2}(|B_n| \ge m_n, |B_{n+1}| < |B_n| + h(|B_n|)) \le Ce^{-cm_n^{(D-1)/D}}. \tag{3.3}$$

Moreover $h(r) \ge cr^{(D-1)/D}/\log r$ for large r. Comparing the recursion for m_n with $x' = cx^{(D-1)/D}/\log x$ gives

$$m_n \ge c\left(\frac{n + M^{1/D}}{\log(n + M + 2)}\right)^D. \tag{3.4}$$

For every $\eta \in \{0,1\}^n$ and $i \in \{0,1\}$, the intrinsic-distance identity and (3.3) show that the probability of

$$|A_n(\eta)| \ge m_n \quad \text{but} \quad |A_{n+1}(\eta i)| < |A_n(\eta)| + h(|A_n(\eta)|)$$

is at most $Ce^{-cm_n^{(D-1)/D}}$. If this failure does not occur, the new vertices carry the extended word and

$$|A_{n+1}(\eta i)| \ge |A_n(\eta)| + h(|A_n(\eta)|) \ge m_{n+1}.$$

By (3.4), when M is sufficiently large,

$$\sum_{n \ge 0} 2^{n+1} Ce^{-cm_n^{(D-1)/D}} < 1. \tag{3.5}$$

Indeed, the negative exponent grows like $(n/\log n)^{D-1}$, which dominates $n \log 2$ because $D > 2$.

Under this conditioning, a union bound over the 2^{n+1} child extensions at level n and (3.5) shows that, with positive probability, no child exploration ever fails. On this event every finite word beginning with 0 is realized. For each infinite word, the finite realizing paths form a finitely branching tree with vertices at every level, so Kőnig's lemma gives an infinite realizing path. The conditioning event has probability $2^{-M} > 0$, so the event that every word beginning with 0 is seen has positive probability. This event is invariant under $\mathrm{Aut}(G)$, and hence has probability one by ergodicity. Exchanging the labels 0 and 1 gives the same conclusion for words beginning with 1. Therefore

$$S_\infty(G) = \{0,1\}^{\mathbb{N}} \qquad \text{almost surely,}$$

and completes the proof of Theorem 1.3. □

How the two proofs fit together.

Both proofs expose the binary tree of finite words and union-bound over its 2^{n+1} transitions at level n. The only difference is what is propagated. On a tree it is the flow-weighted mass from Section 2. On a general graph it is simply the size of the ball in the intrinsic-distance exploration.

Sharpness gives a tail for a large finite cluster. Closing a small next layer turns slow growth of an intrinsic ball into precisely such a cluster, so the same tail controls every word transition. Isoperimetric dimension $D > 2$ makes the resulting comparison

$$2^n \quad \text{versus} \quad \exp\left\{-c\left(\frac{n}{\log n}\right)^{D-1}\right\},$$

and the second term wins because $D - 1 > 1$. This is the entire general-graph argument: intrinsic-ball exploration, the sharpness tail, and one union bound.

We end with the two natural extensions suggested by the proof.

Conjecture 2 (The transitive dimension-two case)**.** *Let G be an infinite connected locally finite transitive graph with $p_c^{\mathrm{site}}(G) < 1/2$ and $\dim_{\mathrm{iso}}(G) = 2$. Then*

$$S_\infty(G) = \{0,1\}^{\mathbb{N}} \qquad \textit{almost surely at } p = \frac{1}{2}.$$

Conjecture 3 (Sharpness above dimension one)**.** *Let G be an infinite connected graph of bounded degree such that $\dim_{\mathrm{iso}}(G) > 1$. Suppose that G satisfies supercritical sharpness at $1/2$ and Bernoulli product measure is ergodic under $\mathrm{Aut}(G)$. Then*

$$S_\infty(G) = \{0,1\}^{\mathbb{N}} \qquad \textit{almost surely at } p = \frac{1}{2}.$$

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