Thm (Kesten, 1980) $p_c(2) = 1/2$.

Lemma (Square root trick)
Let $A_1, A_2 \cdots A_n$ be increasing events with $\mathbb{P}_p(A_1) = \mathbb{P}_p(A_2) = \cdots = \mathbb{P}_p(A_n)$ then

$$\mathbb{P}_p(A_1) \geq 1 - \left(1 - \mathbb{P}_p\left(\bigcup_{i=1}^{n} A_i\right)\right)^{\frac{1}{n}}$$

Pf:

$$1 - \mathbb{P}_p\left(\bigcup_{i=1}^{n} A_i\right) = \mathbb{P}_p\left(\bigcap_{i=1}^{n} A_i^c\right) \geq \prod_{i=1}^{n} \mathbb{P}_p(A_i^c) = (1 - \mathbb{P}(A_1))^n$$

FKG holds for dec events also.

$$\Rightarrow \mathbb{P}_p(A_1) \geq 1 - \left(1 - \mathbb{P}_p\left(\bigcup_{i=1}^{n} A_i\right)\right)^{\frac{1}{n}}$$

∎

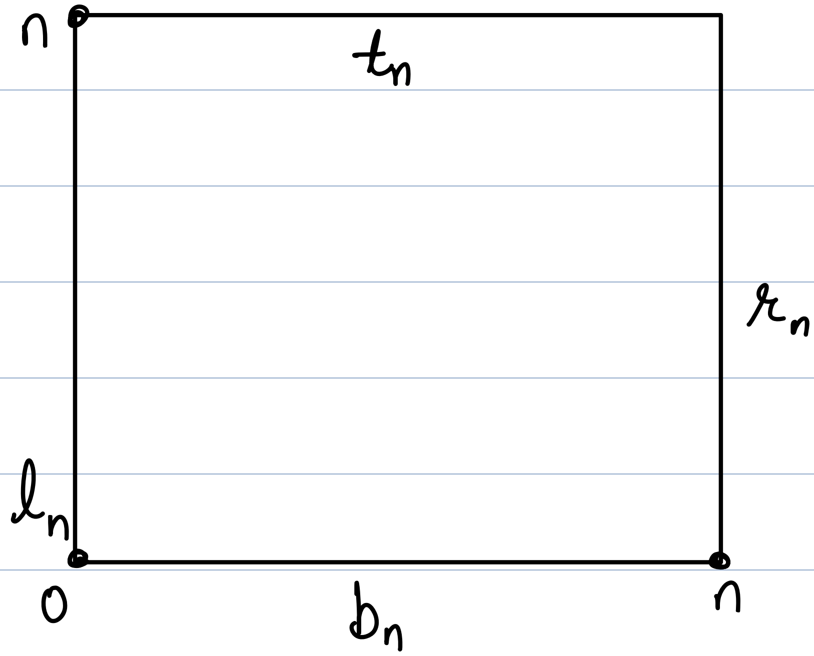
Proof of the theorem

Step ① For $d=2$, $\theta(\frac{1}{2}) = 0 \Rightarrow p_c \geq \frac{1}{2}$

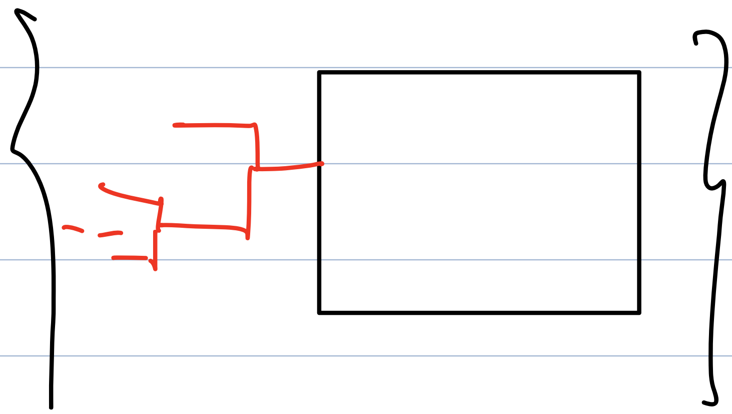
$[0,n]^2$

Let

$L_n = \{\exists$ an infinite open path from a vertex on l_n which does not use any edge from $[0,n]^2\}$



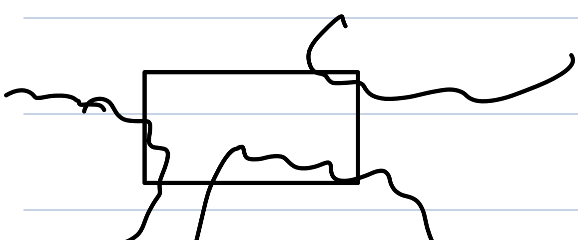
Ill'y define T_n, R_n, B_n

$L_n =$ 

Suppose $\theta(\frac{1}{2}) > 0$, then

$$\mathbb{P}_{\frac{1}{2}}\left(\exists\, u \text{ in } \mathbb{Z}^d \text{ s.t. } C(u) \text{ is unbounded}\right) = 1$$

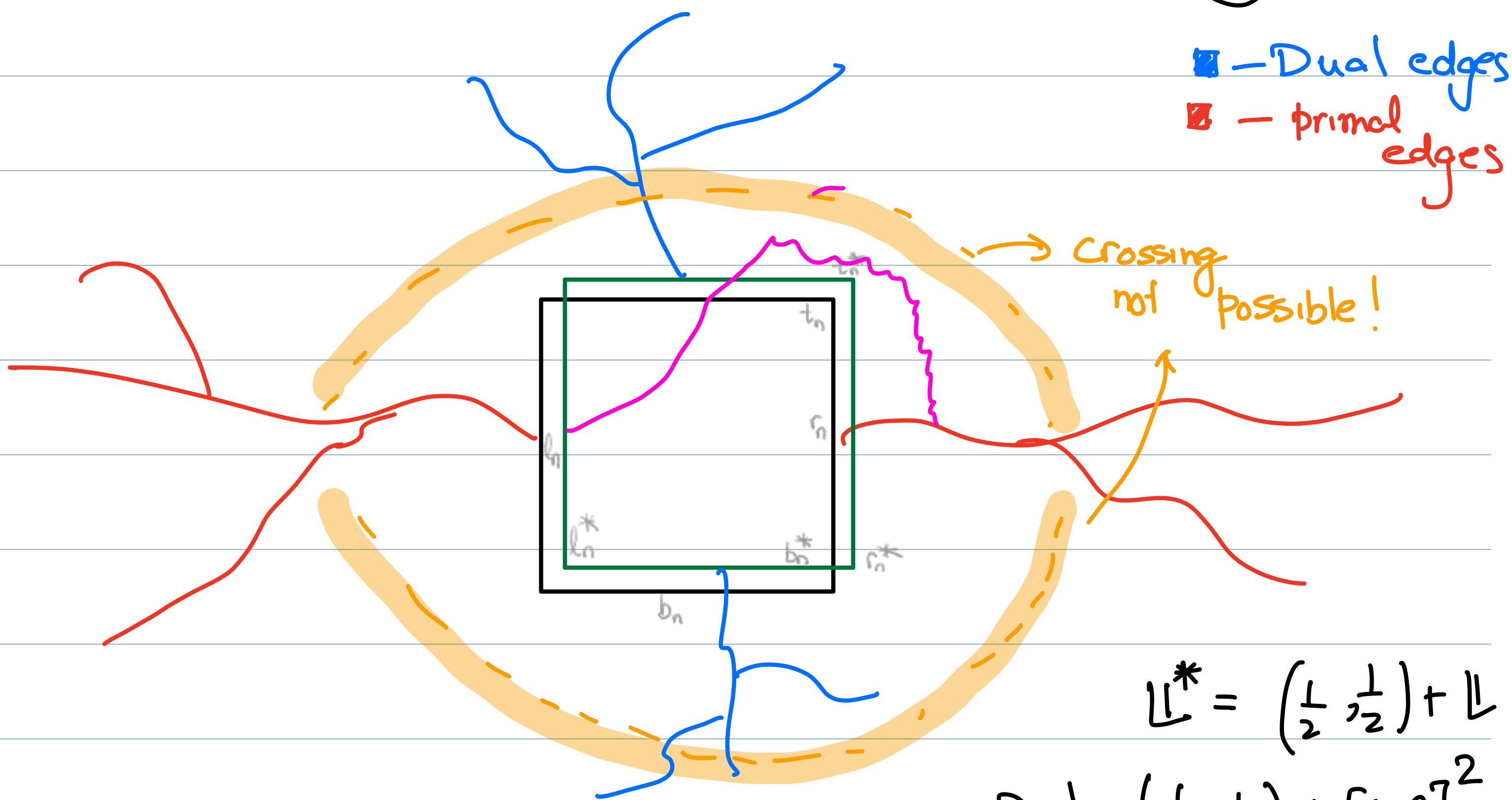
$$\Rightarrow \mathbb{P}_{\frac{1}{2}}\left(L_n \cup R_n \cup B_n \cup T_n\right) \longrightarrow 1 \quad \text{as } n \to \infty$$



$\therefore$ by SRT, $\mathbb{P}_{\frac{1}{2}}(L_n) \geq 1 - \left(1 - \mathbb{P}\left(\bigcup_{i=1} \right)\right)^{\frac{1}{4}}$

$\longrightarrow 1$ as $n \to \infty$

Now $\mathbb{P}(L_n \cap R_n) \longrightarrow 1$ as $n \to \infty$ — (*)



$\mathbb{L}^* = \left(\frac{1}{2}, \frac{1}{2}\right) + \mathbb{L}$

Dual box: $\left(\frac{1}{2}, \frac{1}{2}\right) + [0,n]^2$

$\|^{ly}$ define $\{l_n^*, b_n^*, r_n^*, t_n^*\}$ and $\{T_n^*, B_n^*, L_n^*, R_n^*\}$

i.e. $L_n^* = \left\{ \begin{array}{l} \exists \text{ an unbounded closed path from} \\ l_n^* \text{ which does not use any edge} \\ \text{from } \left(\frac{1}{2}, \frac{1}{2}\right) + [0,n]^2 \end{array} \right\}$

$p = \frac{1}{2}$, the processes on $\mathbb{L}$ and $\mathbb{L}^*$ are exactly the same, Thus by the same argument

$$\mathbb{P}_{\frac{1}{2}}(T_n^* \cap B_n^*) \longrightarrow 1 \quad \text{as } n \to \infty \quad \text{-- } (\mp)$$

From $(*)$ and $(\mp)$,

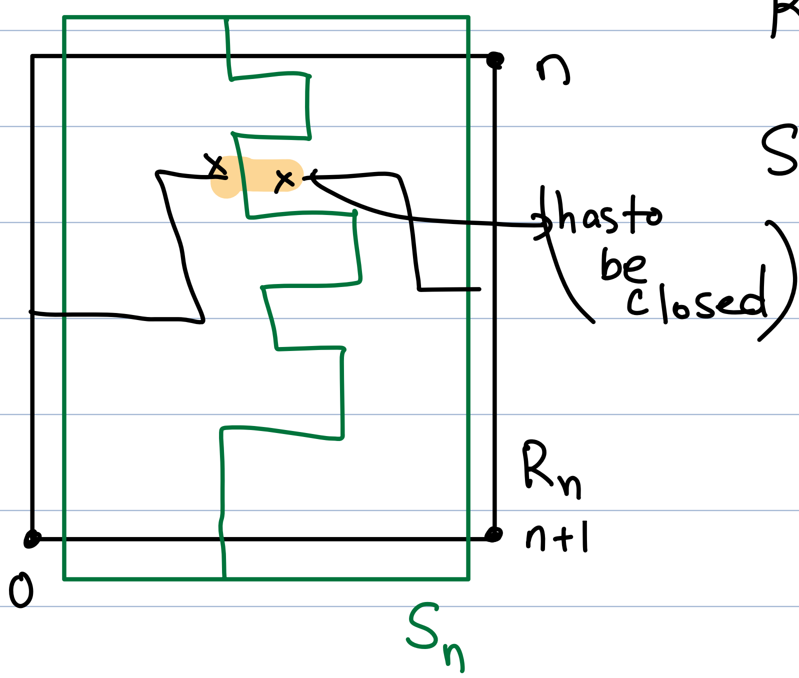
$$\mathbb{P}_{\frac{1}{2}}(L_n \cap R_n \cap T_n^* \cap B_n^*) \longrightarrow 1 \quad \text{as } n \to \infty$$

By uniqueness in $\mathbb{Z}^2$ the two red clusters must be connected by a path not going around, say the pink one. (Because of blue Clusters they cannot be connected around Λ_n)

But due to uniqueness in $(\mathbb{Z}^2)^*$ the two blue clusters must be connected, but due to the red cluster, the path cant go around, and due to the pink path they cannot be connected with help of the box also!

Thus either way we get 2 clusters contradicting uniqueness !!

Step ② $p_c(2) \le \frac{1}{2}$



$R_n = [0, n+1] \times [0, n]$

$S_n = \left(\frac{1}{2}, -\frac{1}{2}\right) + \left([0,n] \times [0, n+1]\right) \subseteq (\mathbb{Z}^2)^*$

$A_n = \{ \exists$ a left-right open crossing of R_n inside $R_n \}$

$B_n = \{ \exists$ a top-bottom closed dual crossing of S_n inside $S_n \}$

Whitney's theorem (graph theory)

Either A_n occurs or B_n occurs not

both.

$$\mathbb{P}_p(A_n) + \mathbb{P}_p(B_n) = 1$$

• {hard part is that one of them must occur}

At $p = \frac{1}{2}$, $\mathbb{P}_{\frac{1}{2}}(A_n) = \mathbb{P}(B_n) = \frac{1}{2}$

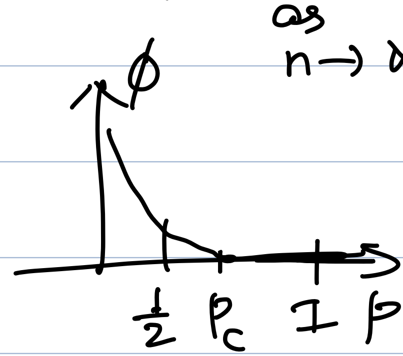
Suppose $p_c > \frac{1}{2}$,

$$\mathbb{P}_{\frac{1}{2}}\{\exists \text{ an open path from } \underset{\sim}{0} \text{ to the line } x = n+1\}$$

$$\leq \mathbb{P}_{\frac{1}{2}}\{C \cap \delta B_{n-1} \neq \phi\}$$

$$\leq \sigma (n-1)^{d-1} e^{-(n-1)\phi(\frac{1}{2})} \longrightarrow 0 \text{ as } n \to \infty$$

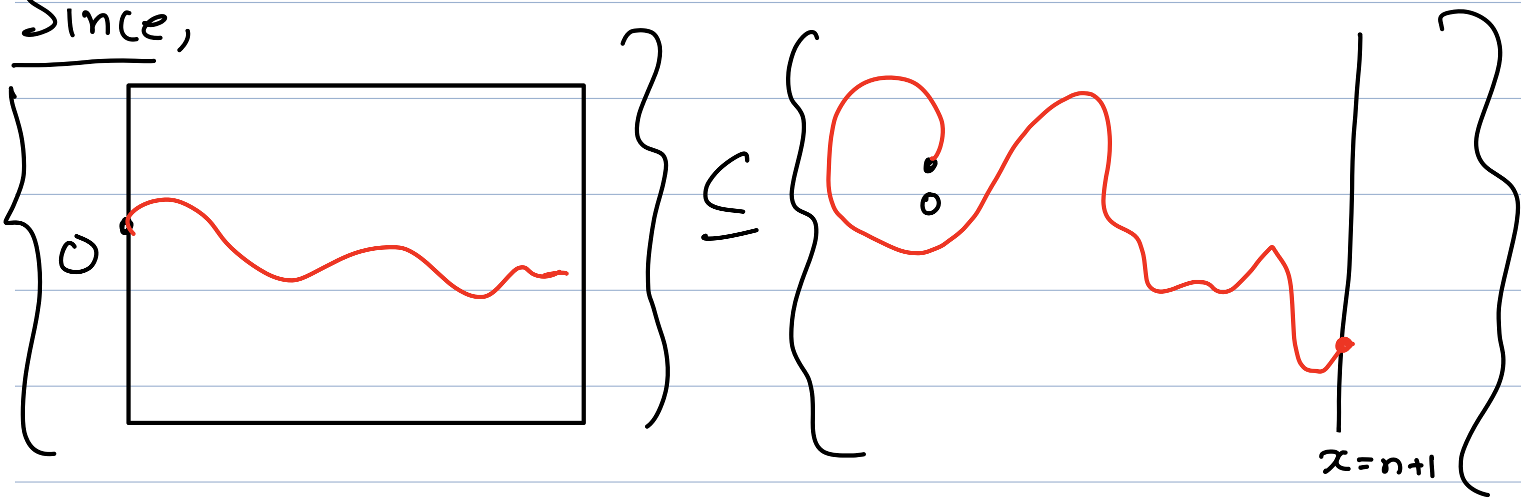
Since $\frac{1}{2} < p_c$, $\phi(\frac{1}{2}) > 0$.



Now by union bound

$$\mathbb{P}_{\frac{1}{2}}(A_n) \leq n\, \sigma\, (n+1)^{d-1}\, e^{-n\phi(\frac{t}{2})}$$

Since,



and there are n points on the left side.

Thus, $\mathbb{P}_{\frac{1}{2}}(A_{\frac{1}{2}}) \longrightarrow 0$ as $n \to \infty$

But $\mathbb{P}_{\frac{1}{2}}(A_n) = 1/2$ $\Rightarrow\Leftarrow$ Thus $p_c(2) \leq \frac{1}{2}$

Therefore we have $\boxed{p_c(2) = \frac{1}{2}}$

Kesten's thm

Let $A_n = \{0 \longleftrightarrow \delta B_n\}$, $B_n = [-n, n]^2$

$\theta_n(p) = \mathbb{P}_p(A_n)$

$\frac{d}{dp}\theta_n(p)$ exists, since A_n depends on finitely many edges.

Duminil – Copin thm

$$\theta_n'(p) \geq C \frac{n\,\theta_n(p)}{\sum_{k=0}^{n-1} \theta_k(p)}$$

Lemma: Let $\{f_n\}_{n\geq 0}$ be a seq of inc and diff'ble functions satisfying

(i) $f_n : (a,b) \longrightarrow (0, M) \quad \forall\, n$

(ii) $\{f_n\}$ converges pointwise in (a,b)

(iii) $f_n' \geq \dfrac{n}{\sum_{k=0}^{n-1} f_k} f_n$

Then $\exists$ $x_0 \in [a,b]$ s.t.

ⓐ $\forall$ $x \in (a, x_0)$ and n large enough s.t.

$$f_n(x) \leq M \exp\left(-\frac{\sqrt{n}\,(x_0 - x)}{2}\right)$$

ⓑ $\forall$ $x \in (x_0, b)$

$$f := \lim_{n\to\infty} f_n \quad \text{satisfies}$$

$$f(x) \geq \frac{x - x_0}{2}.$$