Some Background

Def^n: The seq X_n is called stationary if $(X_n, X_{n+1} \cdots) \overset{d}{=} (X_1, X_2 \cdots\cdots)$

Def^n: An event $A \in \mathcal{B}$ is said to be inv if $\exists$ $B \in \mathcal{B}^{\mathbb{N}}$ s.t. $\forall$ $n \geq 1$

$$A = \{(X_n \cdots) \in B\}$$ and

$$\mathcal{I} = \{A \in \mathcal{B} : A \text{ is invariant}\}$$

$\hookrightarrow$ invariant σ-alg

Thm (SLLN) Let X_i be a stationary seq of random variables s.t. $\mathbb{E}(|X_1|) < \infty$. Let $S_n = \sum_{i=1}^{n} X_i$. Then

$$\frac{S_n}{n} \xrightarrow{a.s.} \mathbb{E}(X_1 | \mathcal{I})$$

Last Class:

Thm The unbounded open cluster, if it exists is unique a.s. $(\mathbb{P}_p)$.

$$(\{0,1\}^{\mathbb{E}}, \mathcal{F}, \mathbb{P}_p)$$

Lemma: Fix $p \in [0,1]$ and $A \in \mathcal{F}$. Given $n \geq 1$ $\exists\, m_n \geq 1$ and an event D_n depending on configuration of the edges in $[-m_n, m_n]^d$ such that

$$\mathbb{P}_p(A \,\Delta\, D_n) \leq \frac{1}{n}.$$

Proof (Bootstrapping + Dynkins $\pi - \lambda$ method)

Let $\mathcal{C} = \{D : D \text{ is a cylinder set}\}$
$= \{D : D \text{ depends on the config of the edges in } B_m \text{ for some } m \geq 1\}$

Let $\mathcal{L} = \{A \in \mathcal{F} : \forall\, n \geq 1\ \exists\, D_n \in \mathcal{C} \text{ s.t. } \mathbb{P}_p(A \,\Delta\, D_n) \leq \frac{1}{n}\}$

Then using Dynkins $\pi-\lambda$ thm we'll show $\mathcal{F} \subseteq \mathcal{L}$.

Exc:

1° $\mathcal{L}$ is a π-system

(1) Non-empty
(2) closed under finite intersections

2° $\mathcal{L}$ is a λ-system

(i) $\Omega \in \mathcal{L}$

Because Ω is a cylinder set

(ii) $A \in \mathcal{L} \Longrightarrow A^c \in \mathcal{L}$, Since $A \Delta B = A^c \Delta B^c$

If $\exists\, D_n$ s.t. $\mathbb{P}_p(A \Delta D_n) \le \frac{1}{n}$

$\Rightarrow \quad \mathbb{P}_p(A^c \Delta D_n^c) \le \frac{1}{n} \Rightarrow A^c \in \mathcal{L}.$

(iii) $\mathcal{L}$ is closed under disjoint union

Fix $n \geq 1$ and $\{A_i : i \geq 1\}$ pairwise disjoint sets in $\mathcal{L}$.

Choose $m = m(n)$ s.t. $\sum_{m+1}^{\infty} \mathbb{P}_p(A_i) < \frac{1}{2n}$

For $i = 1, \dots m$, $A_i \in \mathcal{L}$ to get $D_{n,i} \in \mathcal{C}$ s.t.

$$\mathbb{P}_p(A_i \,\Delta\, D_{n,i}) \le \frac{1}{2nm}$$

$$\mathbb{P}_p\left(\bigcup_{i=1}^{\infty} A_i \,\Delta\, \left(\bigcup_{i=1}^{m} D_{n,i}\right)\right)$$

$$\le \mathbb{P}_p\left(\bigcup_{i=1}^{\infty} A_i \,\Delta\, \bigcup_{i=1}^{m} A_i\right)$$ $[\because A \Delta B \subseteq (A \Delta C) \cup (C \Delta B)]$

$$+ \mathbb{P}\left(\bigcup_{i=1}^{m} A_i \,\Delta\, \bigcup_{i=1}^{m} D_{n,i} \right)$$

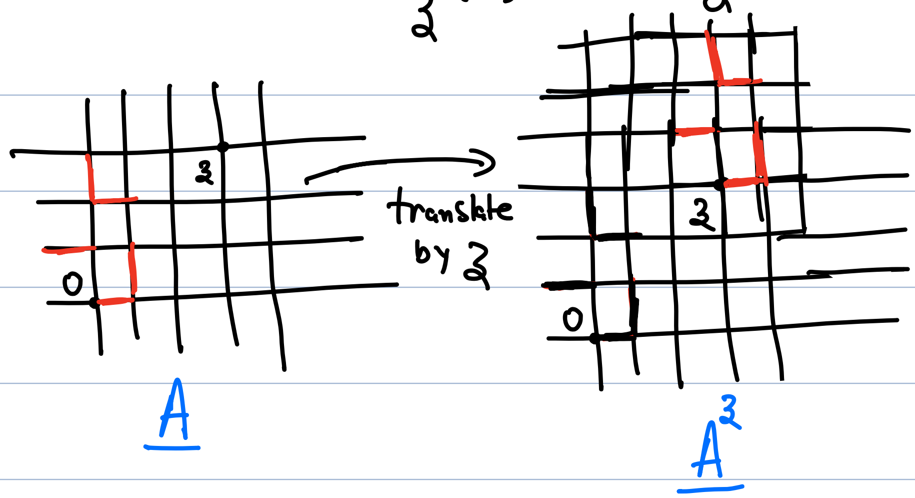
$$\leq \frac{1}{2n} + \frac{m}{2nm} = \frac{1}{n}$$

Since $\mathcal{C} \subseteq \mathcal{L} \Rightarrow \mathcal{F} \subseteq \mathcal{L}$.

$\boxtimes$

For $z \in \mathbb{Z}^d$, and $\omega \in \{0,1\}^E$ define $\omega_z \in \{0,1\}^E$ as follows:

$$\omega_z(e) = \omega(z+e)$$



For an event $A \in \mathcal{F}$ let $A^z = \{\omega : \omega_z \in A\}$

Def^n: A is translation invariant. If $A^z = A$ $\forall z \in \mathbb{Z}^d$

Examples of T.I. events

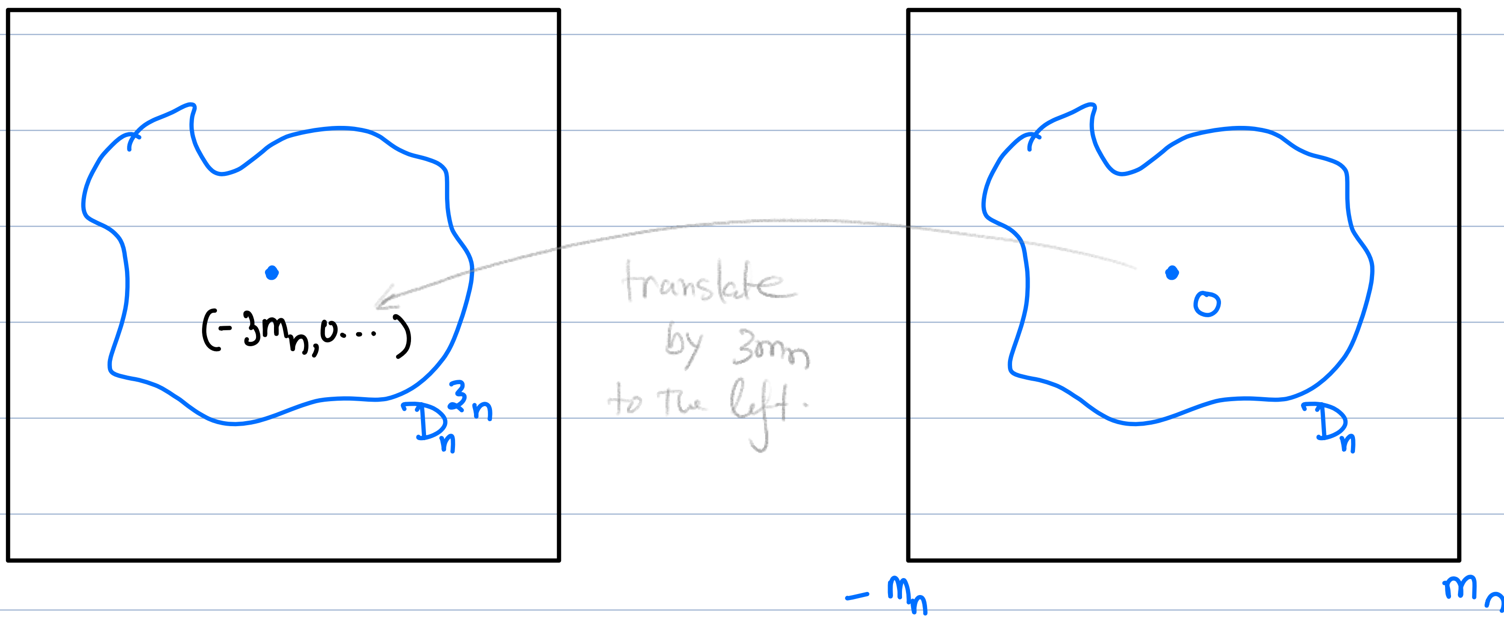
(i) $\{\# C(u) = \infty$ for some $u \in \mathbb{Z}^d\}$

(ii) Let $k \in \mathbb{N} \cup \{\infty\}$ and $E_k = \{\exists$ exactly k disjoint unbounded clusters$\}$

Lemma If A is a T-I. then for any $p \in [0,1]$ we have $\mathbb{P}_p(A)$ is 0 or 1.

Proof: Let D_n be s.t. $\mathbb{P}(A \Delta D_n) \leq 1$

i.e. $\mathbb{P}_p(D_n) \xrightarrow[n\to\infty]{} \mathbb{P}_p(A)$



$D_n^{z_n}$ for $z_n = (-3m_n, 0 \dots)$

$\mathbb{P}_p(D_n \cap D_n^{z_n}) = \mathbb{P}_p(D_n)\,\mathbb{P}_p(D_n^{z_n})$

$\longrightarrow (\mathbb{P}_p(A))^2$

$\mathbb{P}_p(A \Delta D_n^{z_n}) = \mathbb{P}_p(A^{z_n} \Delta D_n^{z_n})$

$$= \mathbb{P}_p(A \Delta D_n) \xrightarrow[\text{as } n\to\infty]{} 0$$

$$\mathbb{P}_p\left(A \Delta (D_n \Delta D_n^{2})\right) \le \mathbb{P}_p(A \Delta D_n) + \mathbb{P}(A \Delta D_n^{2}) \xrightarrow[\text{as } n\to\infty]{} 0$$

$$\Rightarrow \quad \mathbb{P}_p(D_n \Delta D_n^{2n}) \longrightarrow \mathbb{P}_p(A)$$

So $(\mathbb{P}_p(A))^2 = \mathbb{P}_p(A) \Rightarrow \mathbb{P}_p(A) \in \{0,1\}$ $\blacksquare$

Recall:

$E_0, E_1, \dots, E_\infty$ where $E_R = \{\exists$ exactly R disjoint unbounded clusters$\}$

$$\mathbb{P}_p\left(E_0 \sqcup E_\infty \left(\bigsqcup_{i=1}^{\infty} E_i\right)\right) = 1$$

Corollary: $\exists$ $N = N(p) \in \{0,\infty\} \cup \mathbb{N}$ s.t.

$\mathbb{P}_p(E_N) = 1$

$$\mathbb{P}_p(\exists \text{ exactly } N \text{ unbounded open clusters}) = 1$$

For $p \in \{0,1\}$ — trivially done. Assume $p \in (0,1)$.

Step ① : The number of possible unbdd clusters in $0, 1, \infty$

Suppose not then $N \in \{2, 3 \dots\}$

$$\mathbb{P}_p\left(B_m \cap \{\text{all the } N \text{ unbounded clusters}\}\right) \longrightarrow 1 \text{ as } m \to \infty$$

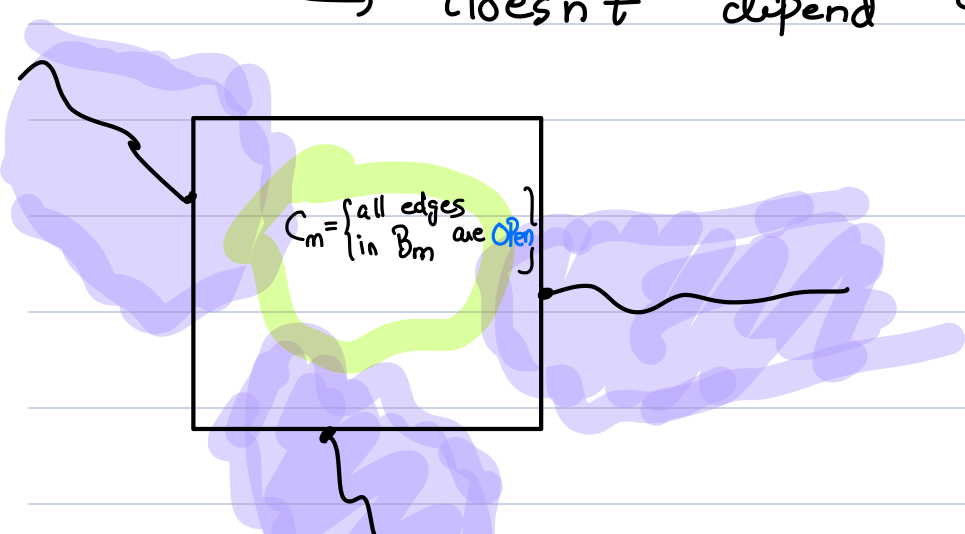
Given $\eta > 0$, $\exists M$ s.t.

$$\mathbb{P}_p\left(B_m \cap \{\text{all the } N \text{ unbounded clusters}\}\right) \geq \eta$$

$\forall \; m \geq M$

Let $A_m := \left\{B_m \cap \{\text{all the } N \text{ unbounded clusters}\}\right\}$

doesn't depend on edges in B_m



$$\mathbb{P}_p(A_m \cap C_m) = \mathbb{P}_p(A_m)\,\mathbb{P}(C_m) \geq \eta\, p^{c m^d} > 0$$

So $\mathbb{P}(E_1) > 0 \Rightarrow \Leftarrow$

! This does not rule out ∞'ly many ∞ clustes because one cannot find a finite Box which intersects all of them.

Step ② Rule out $N = \infty$ (Next time).