Last time: **Lemma:**

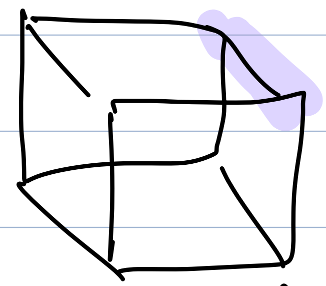
(i) $\beta_{n+m} \leq \#(\delta B_m)\, \beta_n \beta_m$

(ii) $\beta_{n+m} \geq \dfrac{1}{2d\, \#(\delta B_m)} \beta_n \beta_m$

$$\# \delta B_m \leq 2d\,(2m+1)^{d-1}$$

$$= 2d\left(2+\frac{1}{m}\right)^{d-1} m^{d-1}$$

$$\leq d\, 3^d\, m^{d-1}$$



2d sides
each side has $(2m+1)^{d-1}$ vertices

$$\# \delta B_m \leq 2d \# \delta B_m$$
$$\leq d^2\, 3^{d+1}\, m^{d-1}$$

From (i)

$$\log \beta_{n+m} \leq \log \beta_m + \log \beta_n + \log\left(d^2\, 3^{d+1} m^{d-1}\right)$$

From (ii)

$$\log \beta_{n+m} \geq \log \beta_m + \log \beta_n$$

$$-\log\left(d^2 3^{d+1} m^{d-1}\right)$$

Let $g_m := \log\left(d^2 3^{d+1} m^{d-1}\right)$

$$g_m = 2\log d + (d+1)\log 3 + (d-1)\log m$$

$\boxed{\text{WLOG} \quad m \le n}$

Now,

$$g_n + \log \beta_{n+m} \le \log \beta_n + \log \beta_m + g_m + g_n$$

$$= (g_m + \log \beta_m) + (g_n + \log \beta_n)$$

Since,

$$g_n = 2\log d + (d+1)\log 3 + (d-1)\log n$$

$$g_{m+n} = 2\log d + (d+1)\log 3 + (d-1)\log m+n$$

$$g_{m+n} - g_n = (d-1)\log\left(\frac{m+n}{n}\right)$$

Therefore,

$$g_{m+n} + \log \beta_{n+m} \leq g_m + \log \beta_m + g_n + \log \beta_n + (d-1)\log 2$$

$$g_{m+n} + \log \beta_{n+m} + (d-1)\log 2$$

$$\leq \left\{ g_m + \log \beta_m + (d-1)\log 2 \right\} + \left\{ g_n + \log \beta_n + (d-1)\log 2 \right\}$$

Let $a_n = g_n + \log \beta_n + (d-1)\log 2$

Then,

$$\boxed{a_{m+n} \leq a_m + a_n} \quad \text{(Subadditivity)}$$

$\therefore$ Fekete $\Rightarrow$ $\lim\limits_{n\to\infty} \dfrac{a_n}{n}$ (exists) $= \inf\limits_{n\geq 1} \dfrac{a_n}{n}$

i.e. $$\lim_{n\to\infty} \frac{g_n + \log \beta_n + (d-1)\log 2}{n} = \inf_{k\geq 1} \frac{a_k}{k}$$

$$\frac{g_n}{n} \longrightarrow 0 \; ; \quad \lim_{n\to\infty} \frac{\log \beta_n}{n} = \inf_{k \geq 1} \frac{a_k}{k}$$

Define : $\phi(p) := -\lim_{n\to\infty} \frac{\log \beta_n}{n}$ (Rate constant)

We know, $\phi(p) = -\inf_{k\geq 1} \frac{a_k}{k}$

i.e. $\phi(p) \geq \frac{-a_n}{n} \quad \forall \; n \geq 1$

$$a_n \geq -n\,\phi(p) \quad \forall \; n \geq 1$$

$$g_n + \log \beta_n + (d-1)\log 2 \geq -n\,\phi(p) \quad \forall \; n$$

$$\log \beta_n \geq -n\,\phi(p) - g_n - (d-1)\log 2$$

$$= -n\,\phi(p) - \big(2\log d + (d+1)\log 3 + (d-1)\log n\big) - (d-1)\log 2$$

$$= -n\phi(p) - (d-1)\log n - c_1$$

$c_1 = \cdots$ (a constant)

$$\therefore \quad \beta_n \geq C e^{-n\phi(p)} n^{1-d}$$

Exc 1:

Do the same thing using (ii). Taking

$$b_k = g_k - \log \beta_k + (d-1)\log 2$$

Show that: $b_{m+n} \leq b_m + b_n$

and that this implies $g_n - \log \beta_n + (d-1)\log 2 \geq n\phi(p)$

and that this finally implies

$$\log \beta_n \leq -n\phi(p) + (d-1)\log n - c_2$$

$$\Longleftrightarrow \quad \beta_n \le C e^{-n\phi(p)} n^{d-1} \quad \forall n \ge 1$$

Theorem: There exists constants $\sigma, \rho > 0$ and a func $\phi(p)$ s.t.

$$\rho n^{1-d} e^{-n\phi(p)} \le \beta_n \le \sigma n^{d-1} e^{-n\phi(p)}$$

Q.) What is $\phi(p)$?

$$\phi(p) = -\lim_{n\to\infty} \frac{\log \beta_n(p)}{n} \geq 0$$

$$\beta_n(p) = \mathbb{P}_p(0 \longleftrightarrow SB_n)$$

$$\ge \mathbb{P}_p(C(0) \text{ is unbounded})$$

$$= \theta(p) = \begin{cases} >0 & \text{for } p > p_c \\ =0 & \text{for } p < p_c \end{cases}$$

Property 1: $\phi(p) = 0$ for $p > p_c$

Since $\beta_n(p) \leq \sigma\, n^{d-1} e^{-n\phi(p)}$

If $\phi(p) > 0$, then $\beta_n(p) \longrightarrow 0$ as $n \to \infty$

$\Longrightarrow \Longleftarrow$

$$b_n(p) := \frac{-\log \beta_n(p)}{n}$$

$$\log \beta_n \geq -n\phi(p) - (d-1)\log n - c_1$$

$$\log \beta_n \leq -n\phi(p) + (d-1)\log n - c_2$$

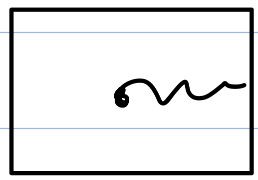
$$\left| n\phi(p) + \log \beta_n \right| \leq C + (d-1)\log n$$

for some $C \geq$

$$|\phi(p) - b_n(p)| \leq \frac{C + (d-1)\log n}{n} \longrightarrow 0 \text{ as } n \to \infty$$

uniform bound $\forall\ p$

i.e. $b_n(p) \longrightarrow \phi(p)$ uniformly in $p \in [0,1]$

$\mathbb{P}_p\Big($  $\Big) = \beta_n(p)$ is a poly in p of degree $\le (2n)^d$

i.e. $b_n(p)$ is continuous, Thus

Property 2 : $\phi(p)$ is continuous in p

Property 3 : $\phi(p_c) = 0$

$$\beta_n(p) \le \mathbb{P}_p(\exists \text{ an open path from the origin of length } \ge n)$$

$$\le 2d(2d-1)^{n-1} p^n \quad \text{(Peierl's Arg)}$$

$$\rho\, n^{1-d} e^{-n\phi(p)} \le \beta_n(p)$$

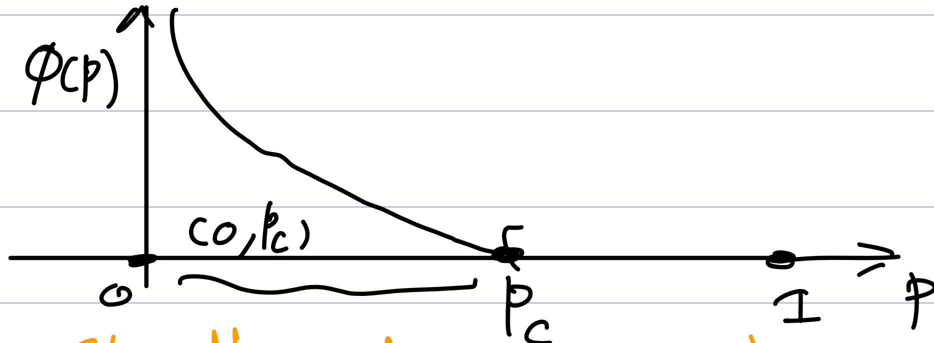
$$\log \rho + (1-d)\log n - n\phi(p) \le \log \beta_n(p) = -n b_n(p)$$

$$\leq n\log(2dp)$$

$$\phi(p) \geq \frac{-n\log 2dp + \log \rho + (1-d)\log n}{n}$$

$$\phi(p) \geq -\log 2dp \longrightarrow \infty \quad \text{as } p \downarrow 0$$

Expectation



Property : $\phi(p)$ is strictly decreasing in $[0, p_c]$

Lemma : For an inc event A depending on the config of finitely many edges only we have

$$\frac{\log \mathbb{P}_p(A)}{\log p} \quad \text{is non-dec in } p$$

Assuming the lemma,

$$\frac{\log \beta_n(p)}{\log p} \leq \frac{\log \beta_n(p')}{\log p'} \quad \text{for } p<p'$$

$$\frac{\frac{1}{n}\log \beta_n(p)}{\log p} \leq \frac{\frac{1}{n}\log(\beta_n(p'))}{\log p'}$$

$n \to \infty$,

$$-\frac{\phi(p)}{\log p} \leq -\frac{\phi(p')}{\log p'}$$

$$\phi(p) \leq \frac{\phi(p')\log\left(\frac{1}{p}\right)}{\log\left(\frac{1}{p'}\right)} < \phi(p')$$

for $p<p'$

Proof of lemma:

Let $h(p) = \mathbb{P}_p(A)$

Claim: $h(p^\gamma) \leq (h(p))^\gamma \quad \forall \; \gamma \geq 1$

Assume the claim:

Let $p' \leq p$, i.e. $p' = p^\gamma$ for some $\gamma \geq 1$

$$\log h(p') = \log h(p^\gamma) \leq \log (h(p))^\gamma$$

$$\leq \gamma \log h(p)$$

$$\frac{\log h(p')}{\underbrace{\gamma \log p}_{\log p'}} \geq \gamma \frac{\log h(p)}{\gamma \log p}$$

$$\Rightarrow \frac{\log h(p')}{\log p'} \geq \frac{\log (h(p))}{\log p}$$

Pf of Claim:

$$h(p^\gamma) \le (h(p))^\gamma \quad \forall \ \gamma \ge 1$$

$h(p) = \mathbb{P}_p(A)$ $\qquad A \uparrow$, finitely dependent

Induction

A depends on m edges $e_1, e_2 \cdots e_m$

$m = 1$ $\qquad A = \begin{cases} \{1\} \longleftarrow h(p) = p \\ \{0,1\} \longleftarrow h(p) = 1 \end{cases}$

So base case is trivially done.

Suppose the result holds for all $k \le m-1$

A depends on edges $e_1, e_2 \cdots e_k \, e_{k+1} \cdots e_m$

$$h(p^\gamma) = \mathbb{P}_{p^\gamma}(A)$$

$$= \mathbb{P}_{p^\gamma}(A \mid \omega(e_1) = 1)\, p^\gamma$$

$$+ \mathbb{P}_{p^\gamma}(A \mid \omega(e_1) = 0)(1 - p^\gamma)$$

$$\leq \Big(\overbrace{\mathbb{P}_p\big(A \mid \omega(e_1)=1\big)}^{x}\Big)^{\gamma} p^{\gamma}$$

(induction)

$$(*) \qquad + \Big(\underbrace{\mathbb{P}_p\big(A \mid \omega(e_1)=0\big)}_{y}\Big)^{\gamma} (1-p^{\gamma})$$

$$= x^{\gamma} p^{\gamma} + y^{\gamma}(1-p^{\gamma})$$

Exc 2: $x \geq y \geq 0$ and $\gamma \geq 1$, Show:

$$x^{\gamma} p^{\gamma} + y^{\gamma}(1-p^{\gamma}) \leq \big(xp + y(1-p)\big)^{\gamma}$$

Since A is inc,

$$\mathbb{P}_p\big(A \mid \omega(e_1)=0\big) \leq \mathbb{P}_p\big(A \mid \omega(e_1)=1\big)$$

and hence Exc 2 applies and

$\therefore$ By $*$

$$h(p^{\gamma}) \leq \big(xp + y(1-p)\big)^{\gamma}$$

$$= \big(\mathbb{P}_p(A)\big)^{\gamma} = \big(h(p)\big)^{\gamma}.$$

For $p > p_c$ $\exists$ an unbdd open cluster

Thm : $\exists$ at most one unbdd open cluster w.p. 1

$$\# \text{ unbdd Clusters } = \begin{cases} 1 & \text{for } p > p_c \\ 0 \text{ or } 1 & \text{for } p = p_c \\ 0 & \text{for } p < p_c \end{cases}$$