Last time: Pivotal edges, Russo's theorem

Proof of Russo's Formula

$(\{0,1\}^E, \mathcal{F}, \mathbb{P}_p)$, $p \in [0,1]$. For each edge $e \in E$ take $p(e) \in [0,1]$ and let $\underset{\sim}{p} = (p(e) : e \in E)$

Let $\{U_e : e \in E\}$ be a collection of iid uniform $[0,1]$ r.v. defined on $(\boxed{H}, \mathcal{G}, \mathbb{P})$

Define $\eta_{\underset{\sim}{p}} \in \{0,1\}^E$ as the config

$$\eta_{\underset{\sim}{p}}(e) = \begin{cases} 1 & \text{if } U_e \le p(e) \\ 0 & \text{o.w.} \end{cases}$$

$(\{0,1\}^E, \mathcal{F}, \mathbb{P}_{\underset{\sim}{p}})$ under $\mathbb{P}_{\underset{\sim}{p}}$ we have

$$\mathbb{P}_{\underset{\sim}{p}}(\eta_{\underset{\sim}{p}} : \eta_{\underset{\sim}{p}}(e) = 1) = p(e)$$

Define $\underset{\sim}{p}'$ from $\underset{\sim}{p}$ as follows: fix an edge f

$$\underset{\sim}{p}'(e) = \begin{cases} p(e) & \text{if } e \neq f \\ p'(f) & \text{if } e = f \end{cases}$$

$\mathbb{P}_{\underset{\sim}{p}}$, $\mathbb{P}_{\underset{\sim}{p}'}$, $\underset{\sim}{p}' = (p'(e) : e \in E)$

$\eta_{\underset{\sim}{p}} \sim \mathbb{P}_{\underset{\sim}{p}}$, $\eta_{\underset{\sim}{p}'} \sim \mathbb{P}_{\underset{\sim}{p}'}$

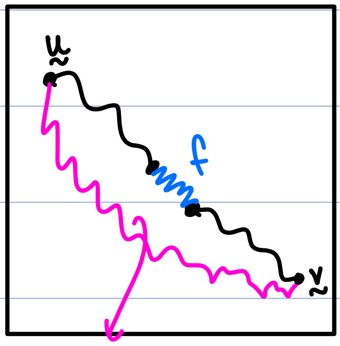
If $p'(f) > p(f)$

$$\mathbb{P}_{\underset{\sim}{p}'}(A) - \mathbb{P}_{\underset{\sim}{p}}(A) = \mathbb{P}\left(\eta_{\underset{\sim}{p}'} \in A \text{ but } \eta_{\underset{\sim}{p}} \notin A\right)$$

$$= \mathbb{P}\left(\{U_f \in (p(f), p'(f)]\} \text{ and } \{f \text{ is pivotal for } A\}\right)$$

depends only on the config of edge f.

depends on everything outside f.



Cannot exist since f is pivotal.

$$= (p'(f) - p(f))\,\mathbb{P}_{\underset{\sim}{p}}(f \text{ is pivotal for } A)$$

If $p'(f) < p(f)$

$$\mathbb{P}_{\underset{\sim}{p}}(A) - \mathbb{P}_{p}(A) = (p(f) - p'(f))\,\mathbb{P}_{\underset{\sim}{p}}(f \text{ is pivotal}).$$

$\mathbb{P}_{p}(\{\omega : f \text{ is pivotal for } (A, \omega)\})$

$$\frac{\delta}{\delta p(f)} \mathbb{P}_{\underset{\sim}{p}}(A) = \mathbb{P}_{\underset{\sim}{p}}(f \text{ is pivotal for } A)$$

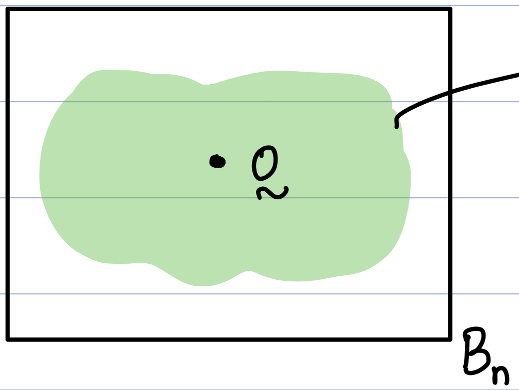
$$\frac{d}{dp} \mathbb{P}_{\underset{\sim}{p}}(A) = \sum_{f \in F} \frac{\delta}{\delta p(f)} \mathbb{P}_{\underset{\sim}{p}}(A)$$

$$= \sum_{f \in F} \mathbb{P}_p(f \text{ is pivotal for } A)$$

$$= \mathbb{E}_{\underset{\sim}{p}}(N(A)) = \mathbb{E}_p(N(A))$$

$\blacksquare$

$B_n = [-n, n]^d$, δB_n = vertices on the boundary of B_n



$\longrightarrow C = \underset{\sim}{C}(0)$

$$\mathbb{P}_p(C \cap \delta B_n \neq \phi) = \beta_n(p)$$

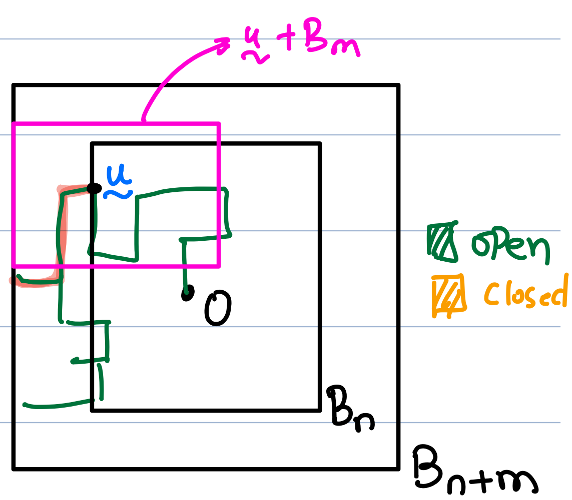
<u>Lemma</u>: (i) $\beta_{n+m} \leq \#(\delta B_n)\, \beta_n \beta_m$

(Choose $n < m$ for better bounds)

(ii) $\beta_{n+m} \geq \dfrac{1}{2d\, \#(\delta B_n)} \beta_n \beta_m$

$C \cap \{\delta B_{n+m}\}$:

$\underset{\sim}{u} \rightarrow$ last vertex connected to 0 through a path in B_n. WRT a path to B_{n+m}



Thus $\{C \cap \{\delta B_{n+m}\}\} \leftarrow$ $\exists$ an open path from $\underset{\sim}{0}$ to $\underset{\sim}{u}$ in δB_n and a (disjoint path from $\underset{\sim}{u}$ to $\delta(\underset{\sim}{u} + B_m)$ outside B_n.)

$A_{m+n}(u)$

$D_m(u)$

(i)

$$\{C \cap \{\delta B_{n+m}\} \neq \phi\} = \bigcup_{\underset{\sim}{u} \in \delta B_n} \left(A_{m+n}(u) \,\square\, D_m(u)\right)$$

$$\beta_{m+n} \leq \sum_{\underset{\sim}{u} \in \delta B_n} \mathbb{P}_p\left(A_{m+n}(u) \,\square\, D_m(u)\right)$$

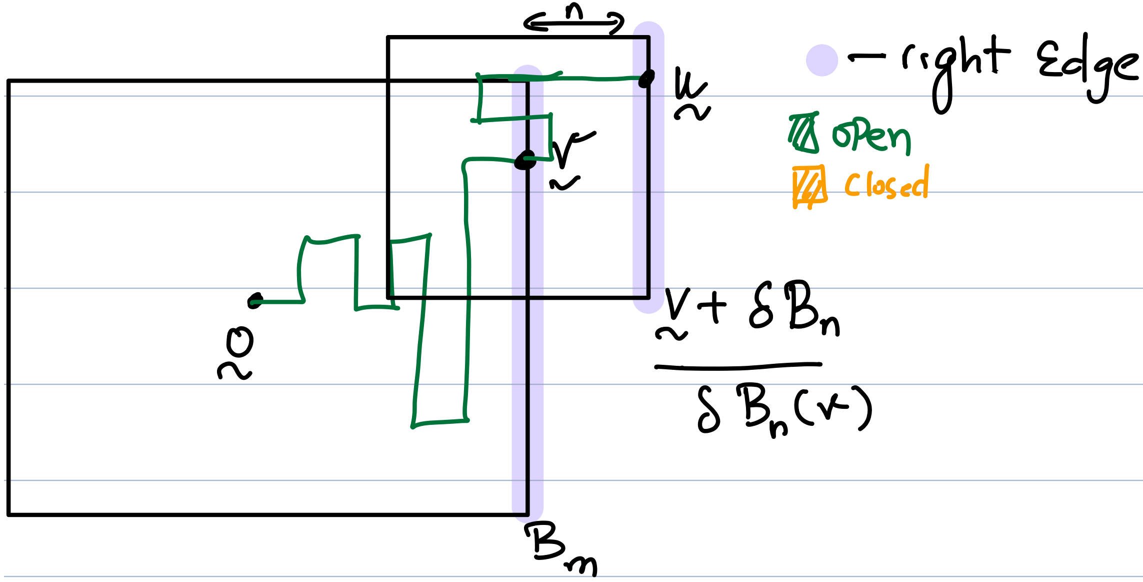
$$\underset{(BK\text{-ineq})}{\leq} \sum_{\underset{\sim}{u} \in \delta B_n} \mathbb{P}_p\left(A_{m+n}(u)\right) \mathbb{P}_p\left(D_m(u)\right)$$

$$\leq \sum_{\underset{\sim}{u} \in \delta B_n} \mathbb{P}_p\left(C \cap \delta B_n \neq \phi\right) \mathbb{P}_p\left(C \cap \delta B_m \neq \phi\right)$$

$$= \left(\# \delta B_n\right) \beta_n \beta_m$$

(ii) $$\beta_{n+m} \geq \frac{1}{2d\, \#(\delta B_n)} \beta_n \beta_m$$

[(i), (ii) can be found in Chp 5 of Grimmett, Eq^ns 5.10]



Fix $v = (v_1, v_2) \in \delta B_m$ with $v_1 = m$ (right side)

$I_v = \{ 0 \longleftrightarrow v$ lying in $B_m \}$

$O_v = \{ v \longleftrightarrow u = (u_1, u_2) \in \delta B_n(v)$ with $u_1 = m+n$ lying inside $B_n(v) \}$

Trivially

$$\{ C \cap \delta B_{n+m} \neq \emptyset \} \supseteq I_v \cap O_v$$

$$\therefore \quad \mathbb{P}_p (C \cap \delta B_{n+m} \neq \emptyset) \geq \mathbb{P}_p (I_v) \, \mathbb{P}_p (O_v)$$

Let $R = \{ 0 \longleftrightarrow w \in B_n$ with $w_1 = n \}$

$T = \{ 0 \longleftrightarrow w \in B_n$ with $w_2 = n \}$

$L = \{ 0 \longleftrightarrow w \in B_n$ with $w_1 = -n \}$

$B = \{ \underset{\sim}{0} \longleftrightarrow \underset{\sim}{\omega} \in B_n \text{ with } \omega_2 = -n \}$

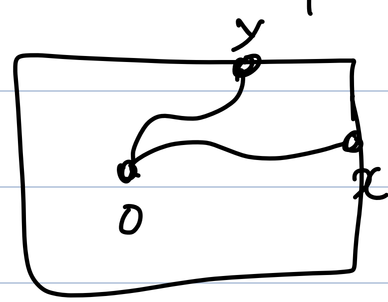
$P(R) = \cdots = P(B)$

$\mathbb{P}_p(0_v) = \mathbb{P}_p(R) \leq \beta_n$

$$\beta_n = \mathbb{P}_p(R \cup T \cup L \cup B)$$
$$\leq 4\mathbb{P}_p(R)$$
$$= 2d\,\mathbb{P}_p(0v) \qquad \text{(in general)}$$

so $\beta_n \geq \mathbb{P}_p(0v) \geq \frac{1}{2d} \beta_n$

$$\beta_m = \mathbb{P}_p\Big(\bigcup_{x \in \delta B_m} \{ \underset{\sim}{0} \longleftrightarrow \underset{\sim}{x} \} \Big)$$
$$\leq \sum_{x \in \delta B_m} \mathbb{P}_p(\underset{\sim}{0} \longleftrightarrow \underset{\sim}{x}) = \# \delta B_m \, \mathbb{P}_p(\underset{\sim}{0} \longleftrightarrow x)$$



So $\exists\, x \in \delta B_m$ s.t.

$$\mathbb{P}_p(\underset{\sim}{0} \longleftrightarrow x) \geq \frac{1}{\# \delta B_m} \beta_m$$

Because of rotational invariance assume $x_1 = m$

Take $\underset{\sim}{x}$ to be $\underset{\sim}{v}$

$$\beta_{n+m} \geq \frac{1}{2d\#\delta B_m} \beta_m \beta_n$$