Lecture 4

Claim: For inc events A and B s.t.

$$A \square B = \left\{ a+b : a \in A, b \in B \ \& \ a_i b_i = 0 \text{ for all } i = 1, 2 \cdots n \right\}$$

Proof of claim:

① **LHS $\subseteq$ RHS**

$c \in A \square B$, by def^n of $A \square B$ $\exists$ disjoint sets $F_1, F_2 \subseteq \{1, 2 \cdots n\}$ such that taking $a, b \in \{0,1\}^n$ with $a_i = \begin{cases} c_i & \text{if } i \in F \\ 0 & \text{o.w.} \end{cases}$

$$b_i = \begin{cases} 0 & \text{if } i \in F_1 \\ c_i & \text{o.w.} \end{cases}$$

Then $a \in A$, $b \in B$, $a_i b_i = 0$ and

$a_i + b_i = c_i \quad \Rightarrow \underline{a+b=c}$

② RHS $\subseteq$ LHS

Let a, b be s.t. $a \in A$, $b \in B$ and $a_i b_i = 0$ $\forall i = 1, 2 \ldots n$. Let $F_1 = \{i : a_i = 1\}$

$F_2 = \{i : b_i = 1\}$

$F_1 \cap F_2 = \emptyset$

A is an inc event, for c ($\in \{0,1\}^n$) with $c_i = 1$ $\forall i \in F_1$, $c \in A$,

$\|^{ly}$ for $d \in \{0,1\}^n$ with $d_i = 1$ $\forall i \in F_2$ $\Rightarrow d \in B$.

This shows the claim.

$\{0,1\}^n$, $\mathbb{P}_p$ on $\{0,1\}^E$

The measure on $\{0,1\}^n$ is P_n The product of n Bernoulli (p) measures with marginals taking values 1 and 0 wp p and $1-p$ resp.

For $A \subseteq \{0,1\}^n$ and $t \in \{0,1\}$ define

$$A_t = \{(a_1, \dots a_{n-1}) \in \{0,1\}^{n-1} : (a_1 \dots a_{n-1}, t) \in A\}$$

$$\subseteq \{0,1\}^{n-1}$$

HW :

(1) For A increasing, A_0 and A_1 are both increasing with $A_0 \subseteq A_1$

(2) Let A, B both increasing events and take $C = A \square B$, use the claim above to Show :

(a) $C_0 = A_0 \square B_0$

(b) $C_1 = (A_0 \square B_1) \cup \underbrace{(A_1 \square B_0)}_{\text{same as } B_1 \square A_0}$

Use question (1) to get

$C_0 \subseteq (A_0 \,\square\, B_1) \cap (A_1 \,\square\, B_0)$

and $C_1 \subseteq A_1 \,\square\, B_1$

(3) For any event A show that

$$P_n(A) = (1-p)P_{n-1}(A_0) + p\,P_{n-1}(A_1)$$

Proof of BK ineq

We'll show this by induction

$n = 1$

There are 3 possibilities

(1) $A = \{0,1\}$ $\quad B = \{0,1\}$ $\quad P_1(A \,\square\, B) = P_1(A)P_1(B) = 1$

(2) $A = \{1\}$ $\quad B = \{0,1\}$
$A = \{0,1\}$ $\quad B = \{1\}$ $\quad P_1(A \,\square\, B) = p$, $P_1(A)P_1(B) = p$

(3) $A = \{1\}$, $B = \{1\}$ $\quad A \,\square\, B = \phi$ so $P_1(A \,\square\, B) = 0 \le P_1(A)P_1(B) = p^2$

BK ineq holds for $n = 1$.

Suppose the BK ineq holds for all $n < m$.

Then,

$$P_{m-1}(C_0) = P_{m-1}(A_0 \square B_0) \overset{\text{Induction}}{\le} P_{m-1}(A_0)P_{m-1}(B_0)$$

(HW 2a)

(HW 2b) $$P_{m-1}(C_1) \le \underbrace{P_{m-1}}_{\text{induction}}(A_1 \square B_1) \le P_{m-1}(A_1)P_{m-1}(B_1)$$

Now,

$$P_{m-1}(C_0) + P_{m-1}(C_1) \le P_{m-1}\big((A_0 \square B_1) \cap (A_1 \square B_0)\big)$$
$$+ P_{m-1}\big((A_0 \square B_0) \cap (A_0 \square B_1)\big)$$

$$\overset{\text{Inclusion-Exclusion}}{=} P_{m-1}(A_0 \square B_0) + P_{m-1}(A_1 \square B_0)$$

$$\overset{\text{induction}}{\le} P_{m-1}(A_0)P_{m-1}(B_0) + P_{m-1}(A_1)P_{m-1}(B_0)$$

$$P_{m-1}(C_0) \le P_{m-1}(A_0)P_{m-1}(B_0)$$

$$\Rightarrow (1-p)^2 P_{m-1}(C_0) \le (1-p)^2 P_{m-1}(A_0)P_{m-1}(B_0)$$

$$P_{m-1}(C_1) \le P_{m-1}(A_1 \square B_1) \le P_{m-1}(A_1) P_{m-1}(B_1)$$

$$\Rightarrow \quad p^2 P_{m-1}(C_1) \le p^2 P_{m-1}(A_1) P_{m-1}(B_1)$$

$$p(1-p)\left(P_{m-1}(C_0) + P_{m-1}(C_1)\right)$$

$$\le p(1-p) P_{m-1}(A_0) P_{m-1}(B_1) + p(1-p)\left(P_{m-1}(A_1) P_{m-1}(B_0)\right)$$

$$P_m(C) = (1-p) P_{m-1}(C_0) + p P_{m-1}(C_1)$$

(add all 3 ineq)

$$\le (1-p) P_{m-1}(A_0)\left((1-p) P_{m-1}(B_0) + p P_{m-1}(B_1)\right)$$

$$+ p P_{m-1}(A_1)\left[p P_{m-1}(B_1) + (1-p) P_{m-1}(B_0)\right]$$

$$= \left[(1-p) P_{m-1}(A_0) + p P_{m-1}(A_1)\right] \times \left[(1-p) P_{m-1}(B_0) + p P_{m-1}(B_1)\right]$$

$$= P_m(A) P_m(B)$$

∎

Suppose A depends on finitely many edges ($\substack{\text{a set} \\ F \text{ say}}$)

$\# F = n$, then $\mathbb{P}_p(A)$ = a poly in p

Also,

$$\mathbb{P}_p(\{\omega\}) = p^{\overset{N(\omega)}{\# \text{ of } 1^s \text{ in } \omega}} (1-p)^{\# 0^s \text{ in } \omega}$$

$$\mathbb{P}_p(A) = \sum_{\omega \in \{0,1\}^n} \mathbb{1}_A(\omega)\, p^{N(\omega)} (1-p)^{n-N(\omega)}$$

— it is differentiable in p.

<u>Thm</u>: For A an event depending on a finite set F of edges of $\mathbb{L}^d$ and $p \in (0,1)$

$$\frac{d}{dp} \mathbb{P}_p(A) = \frac{1}{p(1-p)} \operatorname{Cov}_p(N, \mathbb{1}_A)$$

where N is the number of edges which are open in F. In other words $N(\omega) = \#$ of 1^s in ω.

<u>Pf</u>: $$\frac{d}{dp} \mathbb{P}_p(A) = \sum_{\omega \in \{0,1\}^n} \mathbb{1}_A(\omega) \left[\frac{p^{N(\omega)} N(\omega)}{p} (1-p)^{n-N(\omega)} - p^{N(\omega)} \frac{(1-p)^{n-N(\omega)} (n-N(\omega))}{(1-p)} \right]$$

$$= \sum_{\omega \in \{0,1\}^n} \mathbb{1}_A(\omega)\, p^{N(\omega)} (1-p)^{n-N(\omega)} \left[\frac{N(\omega)}{p} - \frac{(n-N(\omega))}{1-p} \right]$$

$$= \frac{1}{p(1-p)} \underbrace{\sum_{\omega} \mathbb{1}_A(\omega)\, p^{N(\omega)} (1-p)^{n-N(\omega)}} (N(\omega) - np)$$

$$= \frac{1}{p(1-p)} \sum_{\omega} \mathbb{1}_A(\omega)\, N(\omega)\, p^{N(\omega)} (1-p)^{n-N(\omega)} - np\, \mathbb{P}_p(A)$$

$$= \frac{1}{p(1-p)} \left[\mathbb{E}_p(\mathbb{1}_A \cdot N) - \mathbb{E}_p(N)\, \mathbb{E}_p(\mathbb{1}_A) \right]$$

$$= \frac{1}{p(1-p)} \operatorname{cov}(N, \mathbb{1}_A)$$

$\blacksquare$

A is inc depending on a finite set F of edges

$\frac{d}{dp} \mathbb{P}_p(A)$ — studies $\mathbb{P}_{p+\delta}(A) - \mathbb{P}_p(A)$

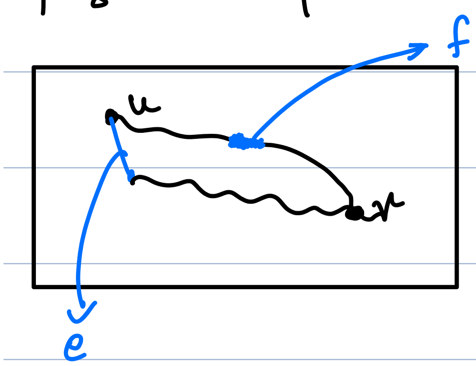
<u>Recall</u>: Coupling $\mathbb{P}_p(A)$, $\mathbb{P}_{p'}(A)$;

$\{U(e) : e \in E\}$ a collection of iid $U(0,1)$ r.v.s on $(\Xi, \mathcal{G}, \mathbb{P})$

For $s \in [0,1]$ let $\eta_s \in \{0,1\}^E$ be the config given by

$$\eta_s(e) = \begin{cases} 1 & \text{if } U_e < s \\ 0 & \text{o.w.} \end{cases}$$

$$\mathbb{P}_{p+\delta}(A) - \mathbb{P}_p(A) = \mathbb{P}_p(\eta_p \notin A, \eta_{p+s} \in A)$$



$\{u \longleftrightarrow v\}$ does not occur.

$U_e \in (p, p+\delta)$ i.e. $\eta_p(e) = 0$, $\eta_{p+\delta}(e) = 1$

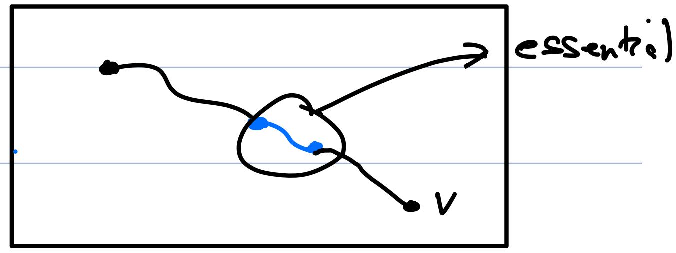
<u>Now</u>,

$$\mathbb{P}(\eta_p(e) = 0, \eta_{p+\delta}(e) = 1) = \delta, \quad \mathbb{P}(\eta_p(f) = 0, \eta_{p+\delta}(f) = 1) = \delta$$

So for both we need $\delta^2 \longrightarrow o(\delta)$

"if δ is very close to 0, the probability of there being two such edges is very small"

So there will be only 1 such edge, and that edge will be essential for the event to occur

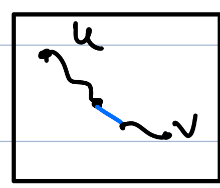


<u>Def^n</u>: Let A be an event and $\omega \in \{0,1\}^E$. An edge $e \in E$ is said to be <u>pivotal</u> for (A, ω) if for a configuration ω' given by

$$\omega'(f) = \begin{cases} \omega(f) & \text{if } f \neq e \\ 1 - \omega(e) & \text{if } f = e \end{cases}$$

we have $\mathbb{1}_A(\omega) \neq \mathbb{1}_A(\omega')$

In particular if A is an inc event, then $\omega \notin A$, $\omega' \in A$. e must be closed under ω and e is open under ω'.



<u>Thm</u> (Russo's pivotal formula)

Let A be an inc event depending on a finite set of edges F. Then

$$\frac{d}{dp} \mathbb{P}_p(A) = \sum_{f \in F} \mathbb{P}_p \{\omega : f \text{ is pivotal for } (A, \omega)\}$$

$= \mathbb{E}_p (N(A))$

where $\underbrace{N(A)}_{\text{r.v.}}$ = (random) number of pivotal edges

we'll show this next class