Lecture 3

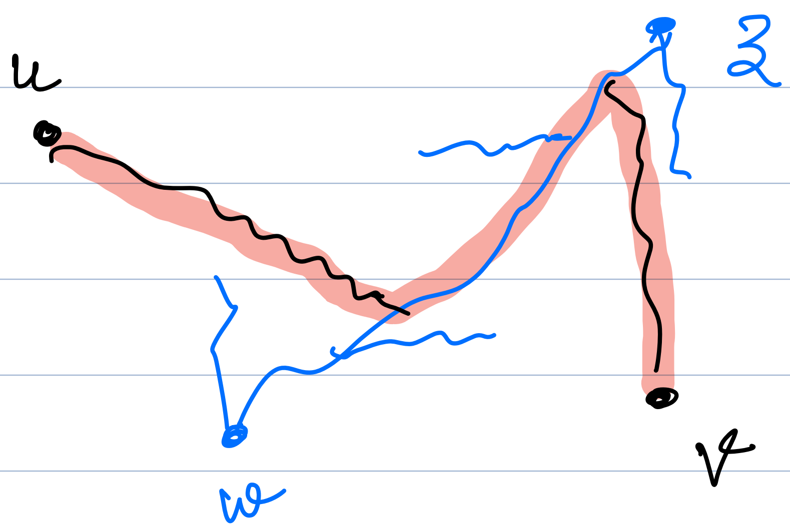
Proof of the FKG ineq

Let $e_1, e_2, \ldots$ be a labelling of all the edges of the lattice $\mathbb{L}^d$.

Case ①: Suppose f_1, f_2 are both inc and they depend on the configuration of finitely many edges.

Intuition of the FKG inequality

$A = \{u \longleftrightarrow v\}$, $B = \{w \longleftrightarrow z\}$

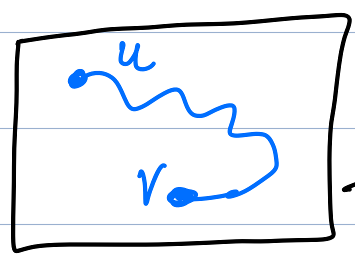


$$\mathbb{P}_p(A \mid B) \geq \mathbb{P}_p(A) \Rightarrow \mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A)\,\mathbb{P}_p(B)$$

A function depending on the configuration of finitely many edges.

$f = \mathbb{1}_A \longrightarrow$ Doesn't depend on finitely many edges

But, if $f = \mathbb{1}_{\{\exists \text{ an open path from } u \text{ to } v \text{ which lies completely in } [-n,n]^d\}}$

$= \mathbb{1}$ 

$\longrightarrow B$

$\hookrightarrow$ depends on only finitely many edges.

If ω and ω' are s.t. $\omega|_B = \omega'|_B$

the $f(\omega) = f(\omega')$.

Back to the proof of case ①

Case ①: Suppose f_1, f_2 are both inc and they depend on the configuration of finitely many edges (say n)

We'll use induction on n.

<u>For $n=1$</u>: f_1, f_2 are function of $\omega(e_1)$ only.

f_1, f_2 are both functions from $\{0,1\} \to \mathbb{R}$

Now take $\eta_1, \eta_2 \in \{0,1\}$

$$f(\eta_1) - f_1(\eta_2) \geq 0 \iff f_2(\eta_1) - f_2(\eta_2) \geq 0$$

$$\Rightarrow \quad \big(f_1(\eta_1) - f_1(\eta_2)\big)\big(f_2(\eta_1) - f_2(\eta_2)\big) \geq 0$$

$$\sum_{\eta_1, \eta_2 \in \{0,1\}} \Big[\big(f_1(\eta_1) - f_1(\eta_2)\big)\big(f_2(\eta_1) - f_2(\eta_2)\big)\Big] \mathbb{P}_p(\omega(e_1) = \eta_1)\, \mathbb{P}_p(\omega(e_1) = \eta_2)$$

$$\geq 0$$

<u>LHS</u>:

$$\sum_{\eta_1=0}^{1} \sum_{\eta_2=0}^{1} f_1(\eta_1) f_2(\eta_1)\, \mathbb{P}(\omega(e_1)=\eta_1)\, \mathbb{P}(\omega(e_1)=\eta_2) + \sum_{\eta_1=0}^{1} \sum_{\eta_2=0}^{1} f_1(\eta_2) f_2(\eta_2)\, \mathbb{P}(\omega(e_1)=\eta_1)\, \mathbb{P}(\omega(e_1)=\eta_2) - \sum_{\eta_1=0}^{1} \sum_{\eta_2=0}^{1} f_1(\eta_1) f_2(\eta_2)\, \mathbb{P}(\omega(e_1)=\eta_1)\, \mathbb{P}(\omega(e_1)=\eta_2) - \sum_{\eta_1=0}^{1} \sum_{\eta_2=0}^{1} f_1(\eta_2) f_2(\eta_1)\, \mathbb{P}(\omega(e_1)=\eta_1)\, \mathbb{P}(\omega(e_1)=\eta_2)$$

$$\mathbb{E}_p(f_1 f_2) + \mathbb{E}_p(f_1 f_2) - \mathbb{E}_p(f_1)\, \mathbb{E}_p(f_2) - \mathbb{E}_p(f_1)\, \mathbb{E}_p(f_2)$$

≥ 0

$$\Rightarrow \quad \boxed{\mathbb{E}_p(f_1 f_2) - \mathbb{E}_p(f_1)\mathbb{E}_p(f_2) \geq 0}$$

Suppose the result holds for $n = 1, 2 \cdots m$ for some $m \geq 1$.
And also suppose that f_1, f_2 are increasing functions depending on the configuration of the edges $e_1, \cdots e_{m+1}$ only.

$$\mathbb{E}_p(f_1 f_2) = \mathbb{E}_p\Big(\mathbb{E}_p\big(f_1 f_2 \mid \omega(e_1) \ldots \omega(e_m)\big)\Big)$$

Apply the case of $n=1$ and say That

$$\mathbb{E}_p\big(f_1 f_2 \mid \omega(e_1), \ldots, \omega(e_m)\big)$$

$$\geq \mathbb{E}_p\big(f_1 \mid \omega(e_1) \ldots \omega(e_m)\big)\, \mathbb{E}_p\big(f_2 \mid \omega(e_1) \ldots \omega(e_m)\big)$$

$\underline{m=2}$ $\quad e_1, e_2, e_3$

f_1, f_2 depend on $\omega(e_1), \omega(e_2), \omega(e_3)$

Fix $\omega(e_1) = \varepsilon_1$, $\omega(e_2) = \varepsilon_2$, $\varepsilon_1, \varepsilon_2 \in \{0,1\}$

$$\mathbb{E}_p\big(f_1 f_2 \mid \omega(e_1) = \varepsilon_1, \ \omega(e_2) = \varepsilon_2\big)$$

$$= \mathbb{E}_p \left(f_1(\varepsilon_1, \varepsilon_2, \omega(e_3)) \, f_2(\varepsilon_1, \varepsilon_2, \omega(e_3)) \right)$$

$$\geq \mathbb{E}_p \left(f_1(\varepsilon_1, \varepsilon_2, \omega(e_3)) \right) \mathbb{E}_p \left(f_2(\varepsilon_1, \varepsilon_2, \omega(e_3)) \right)$$

$$= \mathbb{E}_p \left(f_1 \mid \omega(e_i) = \varepsilon_i \right) \mathbb{E}_p \left(f_2 \mid \omega(e_i) = \varepsilon_i \right)$$

Case (ii): (Infinite dependence) needs martingale Convergence theorem.

* BK Inequality

Suppose A and B are events which depend on a finite set F of edges of $\mathbb{Z}^d$.

$A \square B$ (A box B) – the disjoint occurence of A and B.

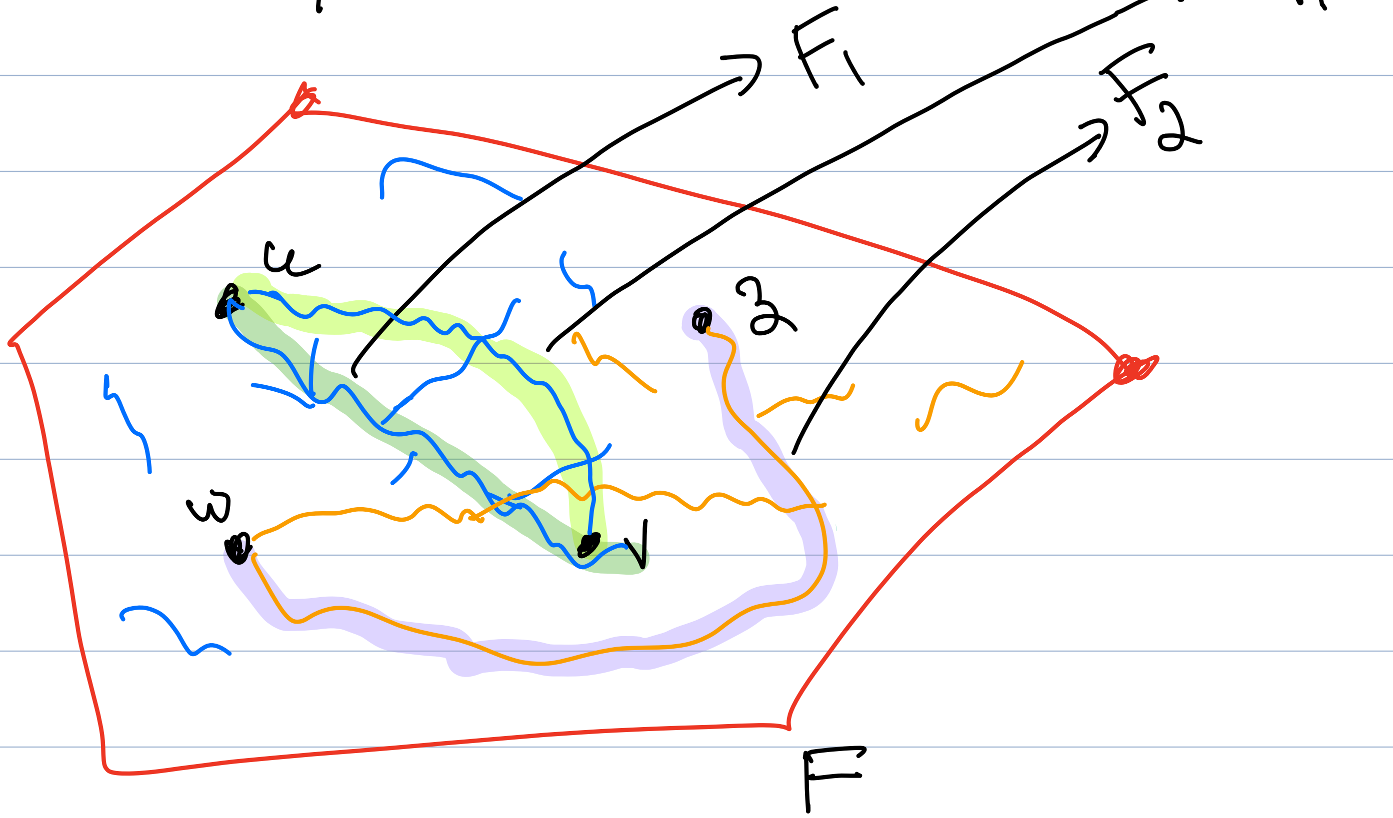
$A \square B :=$ { $\exists$ disjoint sets F_1, F_2 with $F_1 \subseteq F$, $F_2 \subseteq F$ such that the occurence of A depends only on the config of edges in F_1, and the occurence of B depends only on the config of F_2 }

$$A \square B = \left\{ \omega \in \{0,1\}^E : \begin{array}{l} \exists \text{ disjoint sets } F_1 \text{ and } F_2 \\ F_1 \subseteq F,\ F_2 \subseteq F \text{ s.t.} \\ (1)\ \omega_1 \in \{0,1\}^E \text{ with } \omega_1(e) = \omega(e) \text{ for } e \in F_1 \\ \quad \text{then } \omega_1 \in A \\ (2)\ \omega_2 \in \{0,1\}^E \text{ with } \omega_2(e) = \omega(e) \text{ for } e \in F_2 \\ \quad \text{then } \omega_2 \in B \end{array} \right\}$$

Ex

$A = \{u \longleftrightarrow v \text{ in } F\}$

$B = \{w \longleftrightarrow z \text{ in } F\}$



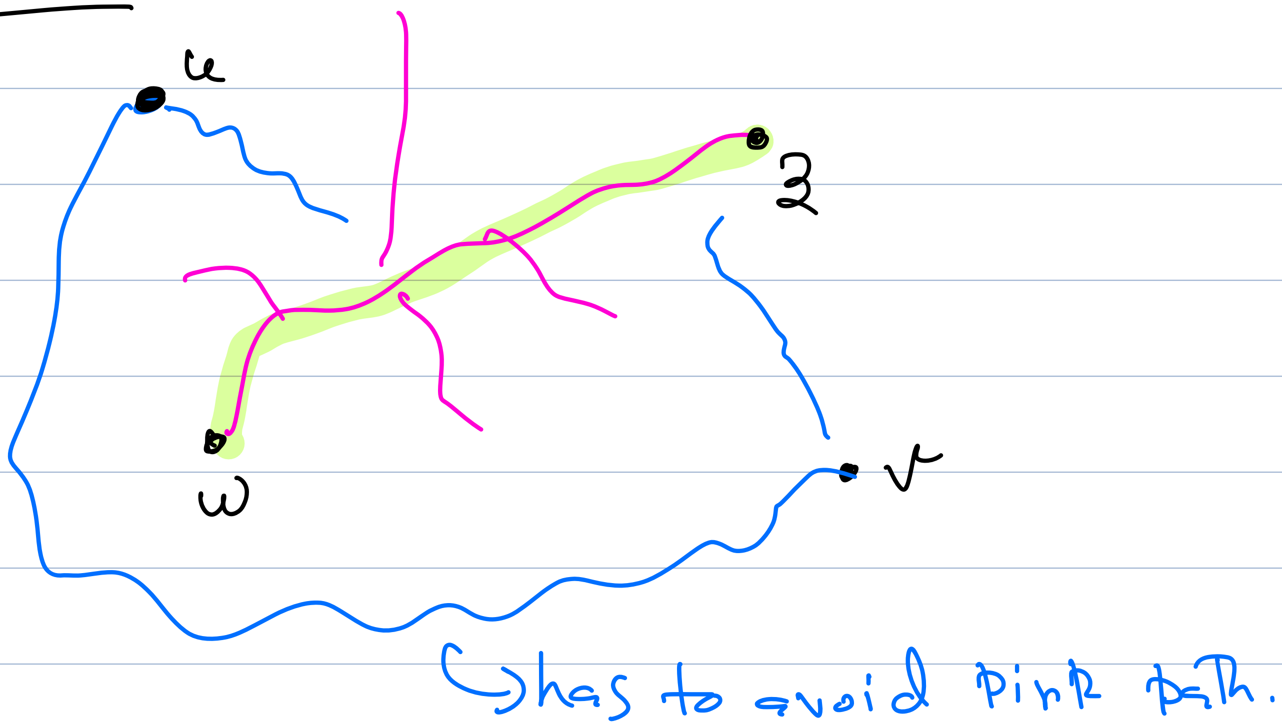
Thm: (BK ineq) Let A and B be both inc events which depend on the config of finitely many edges. Then

$$\mathbb{P}_p(A \square B) \leq \mathbb{P}_p(A)\mathbb{P}_p(B)$$

General — Riemer's Ineq

Alternate Proof by B.V. Rao using Sufficient Statistics

Intitution



Representing $A \square B$:

Suppose A and B depend on the finite set of edges $E_n = \{e_1, e_2 \ldots e_n\}$

We can think of A and B as

$$A \subseteq \{0,1\}^{E_n} \quad \text{and} \quad B \subseteq \{0,1\}^{E_n}$$

$A \subseteq \{\omega : \omega(e_i) \in \{0,1\},\ i = 1,2 \cdots n\}$

$(\omega(e_1), \omega(e_2) \cdots, \omega(e_n))$

↳ an n-tuple of 0s and 1s.

$A \subseteq \{0,1\}^n$, $B \subseteq \{0,1\}^n$

Let $a, b \in \{0,1\}^n$, $a = (a_1, \cdots a_n)$

$b = (b_1, b_2 \cdots b_n)$

$a + b = (a_1 + b_1, \cdots, a_n + b_n)$

$a \circ b = (a_1 b_1, a_2 b_2 \cdots a_n b_n)$

<u>Claim:</u> For inc events A and B

s.t.

$$A \square B = \left\{ a + b : a \in A, b \in B \ \& \ a_i b_i = 0 \text{ for all } i = 1,2 \cdots n \right\}$$

Think about this claim for $n = 2,3,4$.

HW: 1. For an inc event and for $0 \leq p \leq p' \leq 1$
Show that
$$\mathbb{P}_p(A) \leq \mathbb{P}_{p'}(A)$$

2. If A is an inc event then A^c is an dec event.

* Use of BK ineq mostly for finite edges

* Infinite Version