Lecture 2:

Some basic facts:

① $p_c(d+1) \leq p_c(d)$

② $p < p_c(d) \Rightarrow \theta(p) = 0$

$p > p_c(d) \Rightarrow \theta(p) > 0$

③ $p = p_c(d)$ we dont have any info

$p_c(d) \equiv p_c$

Thm: For $d \geq 2$, we have

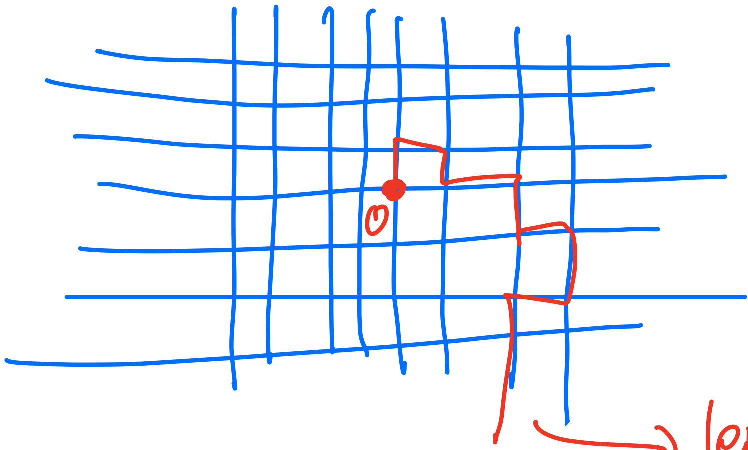
$$0 < p_c(d) < 1$$

Want to show
$$\boxed{\begin{array}{l} p_c(2) < 1 \\ p_c(d) > 0 \quad \forall\, d \geq 2 \end{array}}$$

Actually we'll show $p_c(2) \leq 2/3$

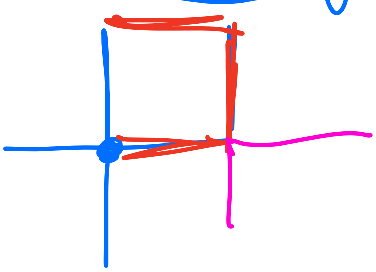
and $p_c(2) \geq \frac{1}{3}$

Proof: ① $p_c(d) \geq \dfrac{1}{2d-1}$



length $\leq n$
Starting from $\underset{\sim}{O}$.

Counting Argument:



First step has $2d$ choices

Next step has $2d-1$ choices

All further step - each has atmost $2d-1$ choices

Total # of paths from the origin of length $\leq n$ is atmost $2d(2d-1)^{n-1}$

For a given path of length n - the prob that the path is open is p^n.

So the expected number of open paths of length n, starting from the origin is

$$\sum_{\substack{\text{over all} \\ \text{paths of} \\ \text{length } n}} p^n \, \mathbb{1}(\text{path is open})$$

$$\leq 2d(2d-1)^{n-1} p^n$$

$$K_n$$

$$\text{Bin}\left(2d(2d-1)^{n-1}, p^n\right)$$

$\therefore$ $\theta(p) \leq \mathbb{P}_p$(there is atleast one open path from the origin of length n)

$$\leq \mathbb{E}_p\left(\begin{matrix}\#\text{ open paths from the}\\ \text{origin of length } n\end{matrix}\right)$$

$$\leq 2d\,(2d-1)^{n-1}\,p^n$$

$$\longrightarrow 0$$

for $p < \frac{1}{2d-1}$

$\therefore\ \theta(p) = 0$ for $p < \frac{1}{2d-1}$

$$\Rightarrow p_c \geq \frac{1}{2d-1}.$$

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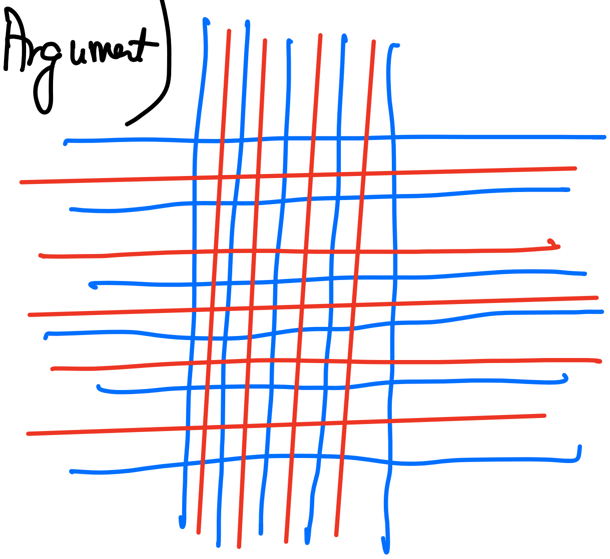
② $p_c(2) \leq \frac{2}{3}$ (Peierls Argument)

$\boxed{d=2}$



Let e^* denote the edge in $\mathbb{L}^*$ which the edge e of $\mathbb{L}$ intersects.

Blue lattice: $\mathbb{L} = \mathbb{Z}^2$

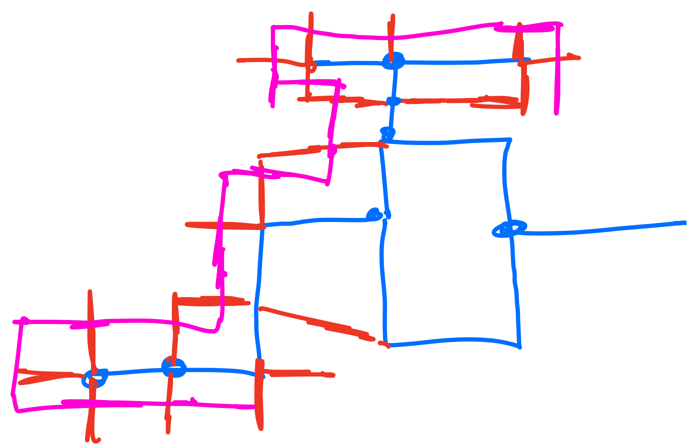
Red lattice: $\mathbb{L}^* = \mathbb{L} + \left(\frac{1}{2}, \frac{1}{2}\right)$

Dual lattice (of $\mathbb{L}$)

The edge e^* is declared open/closed iff the edge e is open/closed.

Suppose the open cluster $C(0)$ of the original lattice is bounded.

closed circuit in the dual lattice



→ Dual Edges

closed edges

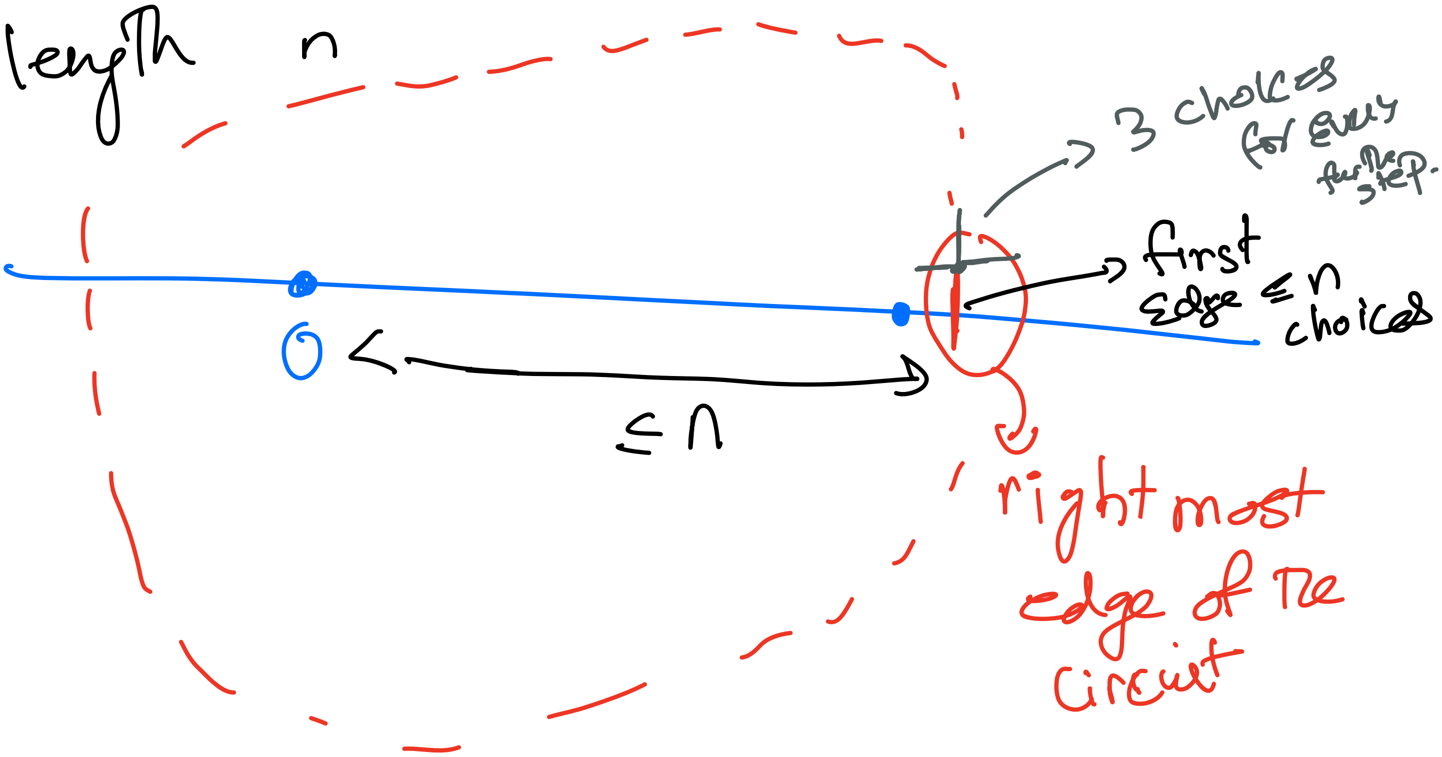
→ open edges.

$\# C(0) < \infty$ iff $\exists$ a closed dual circuit surrounding it

Whitney's Theorem for formalisation.

→ graph theory

Now we'll count the # of circuits in $\mathbb{L}^*$ surrounding the origin of $\mathbb{L}$ of length n



$\therefore$ # closed circuits of length n $\leq n\, 3^{n-1}$

Any such circuit it is closed w.p. $(1-p)^n$

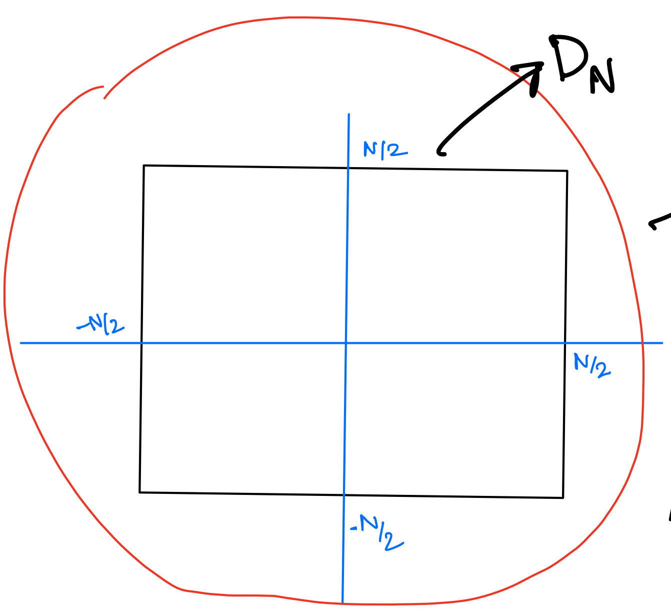
$\mathbb{E}_p$ (# of closed dual circuits surrounding $\underset{\sim}{0}$ of $\mathbb{L}$ of)

$$\left.\begin{aligned} &\leq \sum_{n=1}^{\infty} n\, 3^{n-1} (1-p)^n \\ &< \infty \qquad \text{iff } 1-p < \frac{1}{3} \iff p > \frac{2}{3} \end{aligned}\right\} (*)$$

From (*) for $p > \frac{2}{3}$ Choose N s.t.

$$\sum_{n=N}^{\infty} n\, 3^{n-1} (1-p)^n < \frac{1}{2}$$

For $p > \frac{2}{3}$, $\mathbb{P}_p\left(\nexists \text{ any closed dual circuit of length } N \text{ or more}\right) > 0$



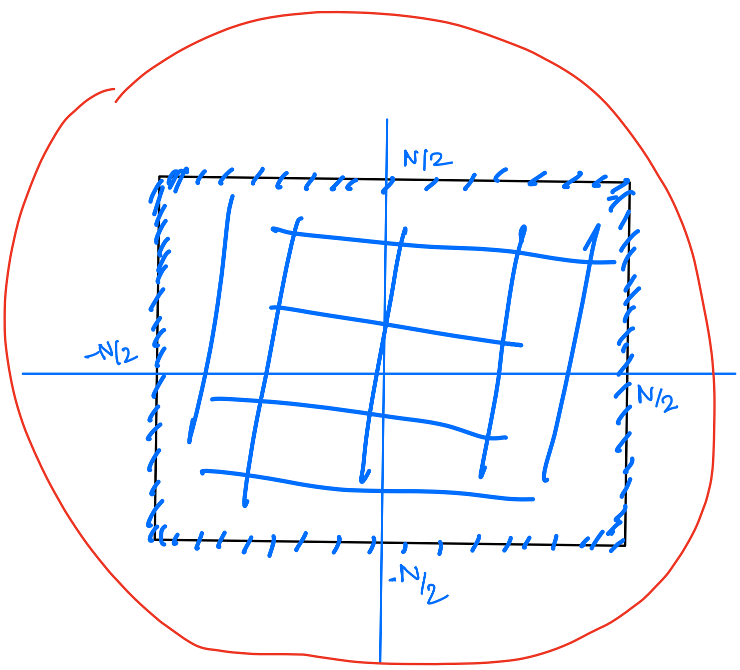
Any dual circuit surrounding the box must have length $\geq N$

$A_n := \{\text{all edges of the lattice } \mathbb{L} \text{ in } D_N \text{ are open}\}$

$B_N := \{\nexists$ any closed dual circuit surrounding $D_N\}$

$\mathbb{P}_p(B_N) > 0$ —— from above, $p > 2/3$

$\mathbb{P}_p(A_N) = p^{N^2} > 0$ for $p > 0$



Event A_N depends on the edges inside the box D_N.

Event B_N depends on the edges of $\mathbb{L}$ outside D_N

A_N and B_N are independent events,

$$\mathbb{P}_p(A_N \cap B_N) = \mathbb{P}_p(A_N)\mathbb{P}_p(B_N) > 0$$

for $p > 2/3$

$\therefore\ \theta(p) > \mathbb{P}_p(A_N)\,\mathbb{P}_p(B_N) > 0$ for

$p > 2/3.$

Such counting arguments in mathematical physics are called Peierls argument — after Rudofl Peierls.

Thus,

$$\boxed{\frac{1}{2d-1} \leq p_c(d) \leq 2/3}$$

Tools (required to study percolation)

Lemma (Subadditive lemma) [Fekete's lemma]

Let $\{a_n : n \geq 1\}$ be a $\mathbb{R}$-valued sequence s.t.

$$a_{m+n} \leq a_m + a_n \quad \forall\ m,n \geq 1$$

Then $\lim_{n\to\infty} \frac{a_n}{n}$ exists and

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geq 1} \frac{a_n}{n}$$

Pf :

① $\liminf_{n} \frac{a_n}{n} \geq \inf_{n \geq 1} \frac{a_n}{n}$

Write $n = \ell m + r$ || $r \in \{0, 1, 2 \dots \ell - 1\}$

by subadditivity,

$$a_n \leq a_{\ell m} + a_r$$

| $a_\ell = \ell + (m-1)\ell$
$a_{\ell m} \leq a_\ell + a_{(m-1)\ell}$

$$\leq m a_\ell + a_r$$

$$\leq m a_\ell + \max\{a_s : s = 0, \dots \ell - 1\}$$

$a_0 = 0$

$$\frac{a_n}{n} \leq \frac{m a_\ell}{n} + \frac{\max\{a_s : s = 0, 1, 2 \dots \ell - 1\}}{n}$$

$$\leq \frac{m a_\ell}{m \ell} + \frac{\max\{a_s : s = 0, 1, 2 \dots \ell - 1\}}{n}$$

$$\longrightarrow \frac{a_\ell}{\ell}$$

as $n \to \infty$

This is true for all $l \geq 1$.

So

$$\frac{a_n}{n} \leq \inf_{l \geq 1} \frac{a_l}{l} \qquad (\forall n)$$

(2)

$$\limsup_{n} \frac{a_n}{n} \leq \inf_{l \geq 1} \frac{a_l}{l}$$

By (1), (2) $\Rightarrow$ Subadditive lemma

Mathematician

<u>FKG Ineq</u> (Harris - FKG ineq)

Physicists

$$\Omega = \{0,1\}^E \qquad \omega, \omega' \in \Omega$$

$\omega(e_1), \omega(e_2), \ldots$

$$\omega : E \to \{0,1\}$$
$$\omega' : E \to \{0,1\}$$

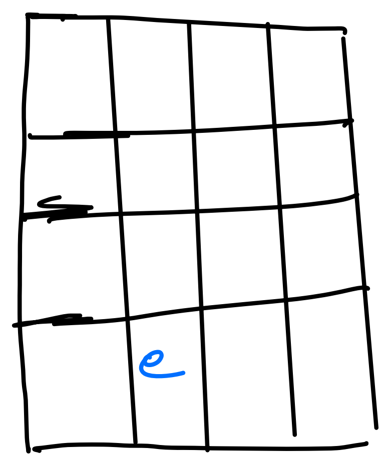
<u>Define a partial order</u>

... if ... $\omega(e) \leq \omega'(e)$

$\omega \le \omega'$ iff $\omega(e) \le$

$\forall\, e \in E.$

If edge e is open in the config ω, Then it is also open in ω'



$\omega(e)=1 \Rightarrow \omega'(e)=1$

$\omega(e)=0 \Rightarrow \omega'(e)=0$ or 1

$\forall\, e\in E$

<u>Defn</u> : A function $f : \Omega \to \mathbb{R}$ is increasing if $f(\omega) \le f(\omega')$ for all $\omega \le \omega'$, dec if $f(\omega) \geq f(\omega')$ for all $\omega \le \omega'$.

$(\Omega, \mathcal{F}, \mathbb{P}_p)$

An event $A \in \mathcal{F}$ is increasing/decreasing if $\mathbb{1}_A$ is inc/dec

<u>Thm</u> (FKG Ineq) Let $f_1, f_2 : \Omega \to \mathbb{R}$ be both increasing or both decreasing.

Assume they are square integrable (i.e. $\int_\Omega f_i^2 \, d\mathbb{P}_p < \infty$ for $i = 1,2$)

<u>Then</u>

$$\mathbb{E}_p(f_1 f_2) \geq \mathbb{E}_p(f_1)\mathbb{E}_p(f_2)$$

In particular if $f_1 = \mathbb{1}_A$ and $f_2 = \mathbb{1}_B$

then,

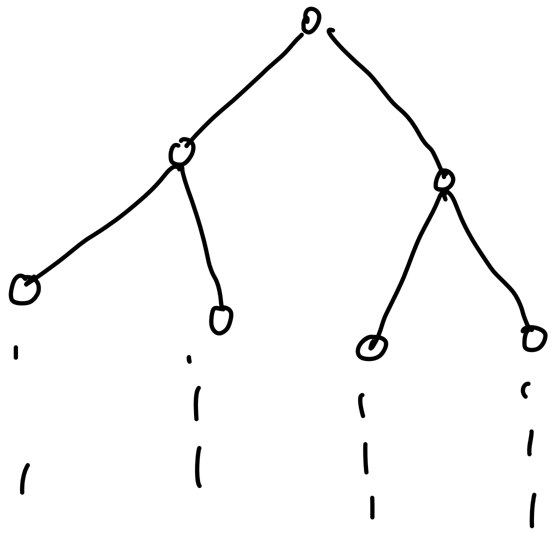
$$\mathbb{P}_p(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B)$$

* FKG tells us that $\mathbb{E}_p(f_1 f_2) - \mathbb{E}_p(f_1)\mathbb{E}_p(f_2) \geq 0$

i.e. $\mathrm{cov}(f_1, f_2) \geq 0$ i.e f_1, f_2 are positively correlated.

Exercise

(1)



$\mathbb{T}$ = a rooted binary tree

- Find $p_c(\mathbb{T})$

- Using Kolmogorov's 0-1 law show that

$$\theta(p) > 0 \Rightarrow \mathbb{P}_p(\exists u \in V \text{ s.t. } \#C(u) = \infty) = 1$$

- $\{u \longleftrightarrow v\}$ is increasing.