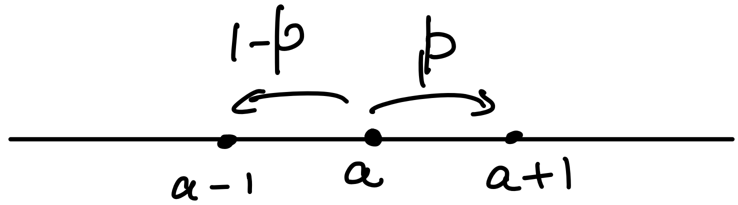
Lecture 1

24th July 2024

- Random walks

$S_0 = a$, $S_n =$ position at time n



Increments : $X_i = \begin{cases} +1 & \text{wp } p \\ -1 & \text{wp } 1-p \end{cases} \quad \forall\ i \geq 1$

Then, $S_n = a + \sum_{i=1}^{n} X_i$. In such a setup, the particle has the randomness inbuilt, the environment/medium is fixed (here, its the line $\mathbb{Z}$)

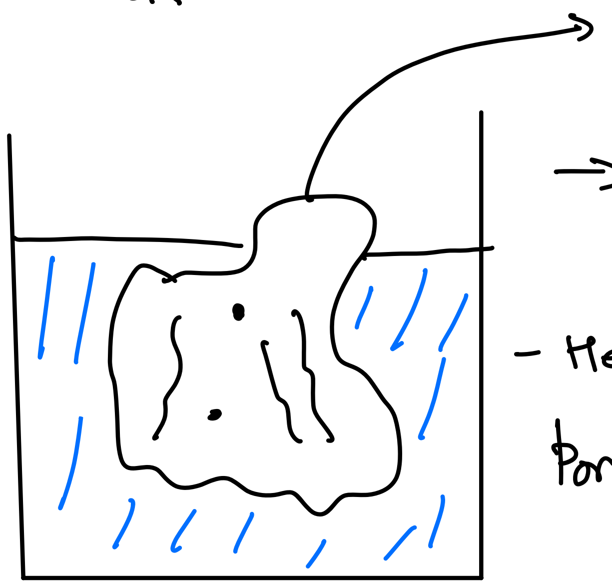
In percolation theory, the medium is random (random subgraph for instance) and the particle is allowed to move freely in the medium.

#1957 : Percolation theory / model introduced by Broadbent and Hammersley

Geologist

Math (applied)

To Study : The flow of liquid in Porous rock



→ Some Pores will get wet while others won't.

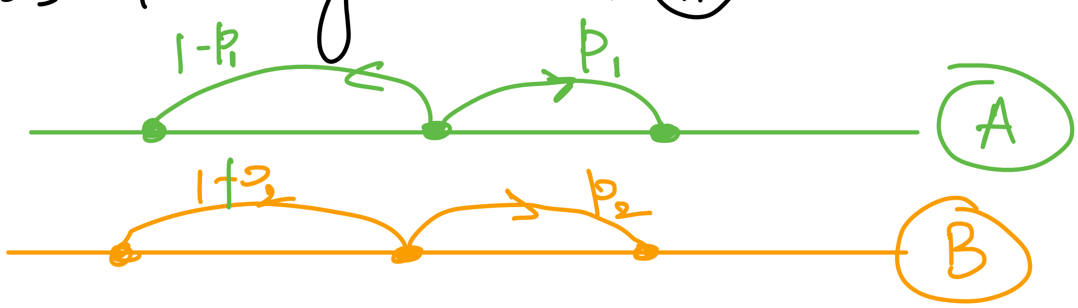
- Here, the medium i.e., the Pore structure is random.

- Granite / marble → Very diff Pore structures
- Sandstone ↗

So if we go back to The RW analogy, we need a Parameter to control this 'porosity'

By the same argument, we may except water to Percolate more in sandstone then in Granite.

For $p_2 > p_1$ B is likely to be more towards the right than A



Goal: ① Bond percolation on $\mathbb{Z}^d$

- Grimmett - Percolation (1999)
- Bollobas + Riordan (2006)

② Orientable percolation

→ Some articles, no books per se

③ Continuum percolation

I Bond Percolation

$G = (V, E)$ a graph, where V is countable (finite / infinite) and E is a collection of unordered pairs of vertices.

In percolation theory, Vertices ≡ sites
edges ≡ bonds

Def^n: A path π is a collection of vertices and edges (finite or infinite)

$(v_1, e_1, v_2 \dots, v_{n-1}, e_n, v_n, \dots)$ where

$e_i = \langle v_i, v_{i+1} \rangle \quad \forall\ i \geq 1, v_i \in V, e_i \in E.$

angular brackets ≡ unordered

- We require paths to be self-avoiding i.e., $v_1, v_2 \cdots v_n$ are all distinct.

- Two vertices u and v are said to be connected by a path if $\exists\ \pi = (v_1 \ldots v_n)$ s.t. $v_1 = u$, and $v_n = v$.

Going back to the analogy of pores - we would like to think of them as pipes. Imagine pipes with a tap in the middle - which can be opened or closed. So, we wish to classify edges as either open or closed.

- We want to carry out this classification in an iid fashion. An edge is

$$\begin{cases} \text{open} & \text{wp } p & [1] \\ \text{closed} & \text{wp } 1-p & [0] \end{cases}$$

labels

Probabilistic formulation : Take $V = \mathbb{Z}^d$ and

$$E = \{ \langle u, v \rangle : \|u - v\|_1 = 1 \}$$

$\to$ or $\|u-v\|_2 = 1$ also works.

This graph will be called the lattice $\mathbb{L}^d = (\mathbb{Z}^d, E)$.

Let (i) $\Omega = \{0,1\}^E \longrightarrow$ Set of all labels

(ii) $\mathcal{F}$ = σ - algebra generated by cylinder sets

of the form

$$\left\{ \omega(e_i) = \varepsilon_i \;\middle|\; \begin{array}{l} e_i \in E, \quad \varepsilon_i \in \{0,1\} \\ 1 \le i \le n, \quad n \geq 1 \end{array} \right\}$$

The elements depend only on finitely many edges.

(iii) $\mathbb{P}_p$ – Product measure, given by its marginals

$$p\,\delta_{\{1\}} + (1-p)\,\delta_{\{0\}}$$

This gives us a prob. space $(\Omega, \mathcal{F}, \mathbb{P}_p)$

Parametrized by p

Product space of Bernoulli r.v.'s

[p for Sandstone $>$ p for Granite]

- Note that the def^n of the $\mathbb{P}$-space doesn't depend on the integer lattice $\mathbb{L}^d$ — can be done for any graph.

Alternatively, we could consider <u>site percolation</u>

- The vertices will be assigned <u>open</u> or <u>closed</u>

$$\Omega = \{0,1\}^V$$

$$\mathcal{F} = \sigma(\text{cylinder sets}) = \sigma\left(\left\{\omega \in \Omega \,\middle|\, \begin{array}{l}\omega(v_i) = \varepsilon_i \in \{0,1\} \\ 1 \le i \le n,\ n \ge 1\end{array}\right\}\right)$$

$\mathbb{P}_p$ — product measure with marginals

$$p\,\delta_{\{1\}} + (1-p)\,\delta_{\{0\}}$$

— We will stick to bond percolation in this course

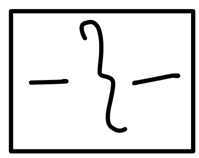
- A path π is open (closed) if all edges in π are open (closed)

- Two vertices u, v are said to be conn. by an open path if $\exists$ an open path connecting these two vertices

<u>Notation</u>: u and v conn $\equiv u \longleftrightarrow v$

Being closed is also an interesting property - useful in many dual arguments

- Will be useful later on, when dealing with the dual lattice



- The <u>open cluster</u> of $u \in V$ is

$$C(u) = \{ v \in V \mid v \longleftrightarrow u \}$$

for $\underset{\sim}{0} \in \mathbb{Z}^d$, let $C := C(0) = \{ v \mid v \longleftrightarrow 0 \}$

<u>Note</u>: C — random set of vertices of $\mathbb{Z}^d$ (or V)

H.W.

① Show that $\{u \longleftrightarrow v\} \in \mathcal{F}$

② $\{\# C \geq 10\} \in \mathcal{F}$

③ $\{\# C < \infty\} \in \mathcal{F}$

In particular we can define,

$$\theta(p) := \mathbb{P}_p(\# C = \infty).$$

Note that as $\mathbb{Z}^d$ is transitive, $\underset{\sim}{0}$ is an arbitrary choice.

Note: (i) $p = 0 \Rightarrow C = \{0\}$ a.s. ; $\theta(p) = 0$

(ii) $p = 1 \Rightarrow C = \{1\}$ a.s. $\Rightarrow \theta(p) = 1$

- We go back to the RW example, as $p \uparrow$, the random walker with larger p proceeds more to the right on avg.

- Here again, we expect $\theta(p_1) \le \theta(p_2)$ when $p_1 \le p_2$. [we expect the cluster of p_1 to be "smaller" than p_2]

Thm ① : If $p_1 \le p_2$, Then $\theta(p_1) \le \theta(p_2)$

Proof: Coupling (Doeblin : 2nd WW)

Eg: Renewal Thm in MC (See Grimmett stirzaker)

Let $U_1, U_2, \dots$ be iid Unif $[0,1]$ r.v. on some prob space $(\Omega, \mathcal{F}, \mathbb{P})$. $\mathbb{L}^d$ has countably many edges - label them $\{e_1, e_2 \dots\}$.

We consider two lattices A and B. Fix $p_1 \le p_2 \in (0,1)$. Let E^A and E^B denote the labelled edge sets of A and B.

Set,

$$\ell(e_i^A) = \mathbb{1}_{U_i \geq 1-p_1}$$

$$\ell(e_i^B) = \mathbb{1}_{U_i \geq 1-p_2}$$

Then the induced measure on A is $\mathbb{P}_{p_1}$ and that on B is $\mathbb{P}_{p_2}$. Thus,

$$\theta(p_1) := \mathbb{P}_{p_1}(|C| = +\infty)$$

$\ell(e_i^A) = 1 \Rightarrow \ell(e_i^B) = 1$ but not vice versa

$$= \mathbb{P}(|C_A| = +\infty)$$

$$\leq \mathbb{P}(|C_B| = +\infty)$$

By construction

$$= \mathbb{P}_{p_2}(|C| = +\infty) = \theta(p_2)$$

(Note: $\mathbb{1}_{U_i \leq p_1}$, $\mathbb{1}_{U_i \leq p_2}$ would also work)

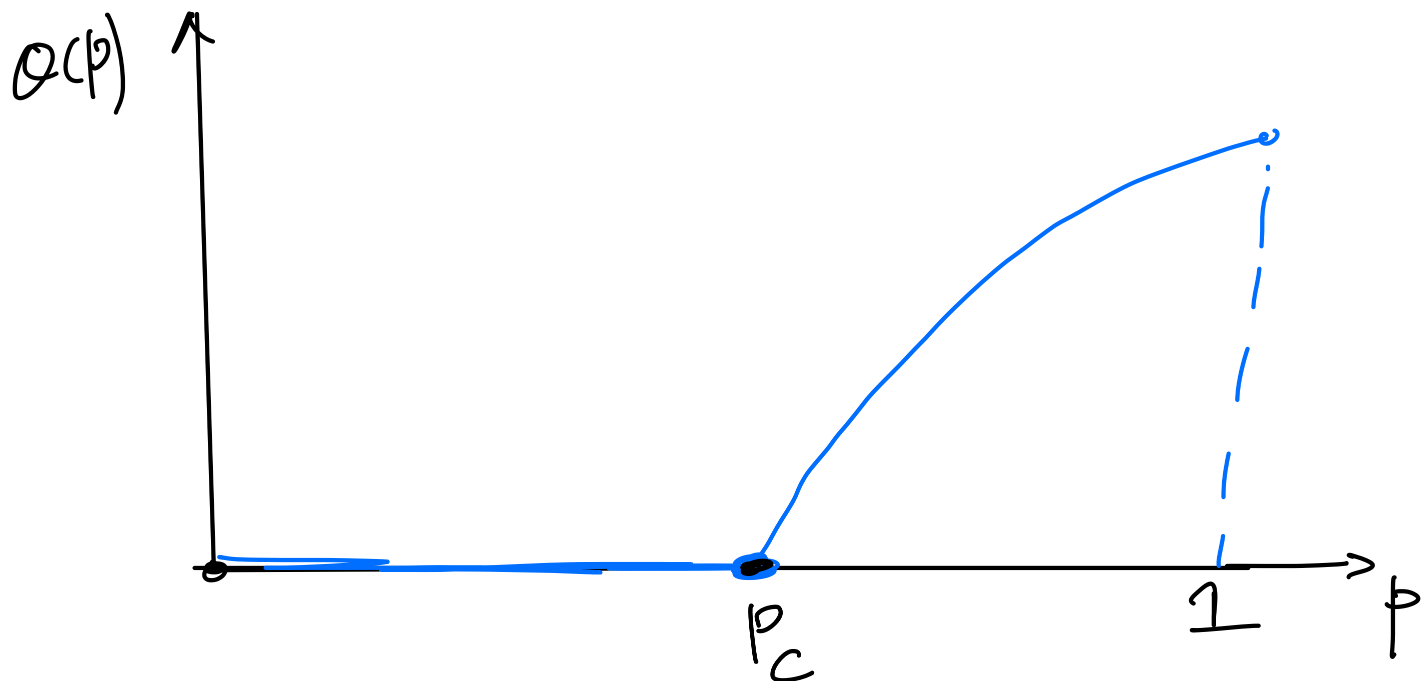
- What we essentially did in coupling:

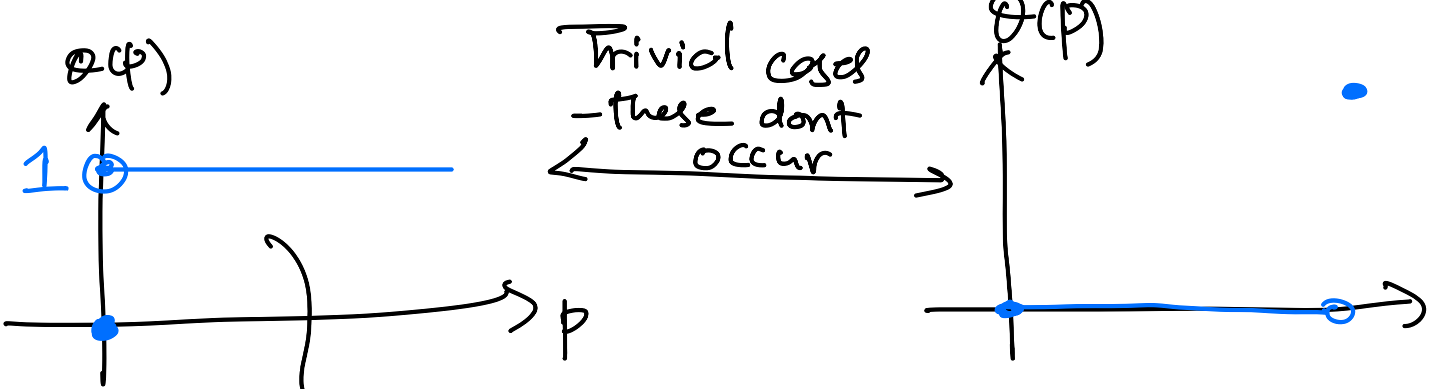
Went to a new $\mathbb{P}$-space, and defined C_A C_B new rv^s there. which have same dist as $C_0|_{p_1}$, $C_0|_{p_2}$ but we could compare them!

<u>What we have shown</u> : $\theta(p)$ is monotonic non-decreasing.

<u>Def^n</u>: The critical parameter of bond percolation on $\mathbb{Z}^d$ is defined as

$$p_c(d) = \inf\{p \mid \theta(p) > 0\}$$





HW: for $d = 1$

it is a triviality