Continuum Percolation

Lecture-19

(Homogeneous)

Poisson point process:

On $\mathbb{R}^d$, we define a process $X = (x_1, x_2, \dots)$, $x_i \in \mathbb{R}^d$ which satisfies

i) For $A \subseteq \mathbb{R}^d$ Borel m'ble with ($\ell \equiv$ Lebesgue m'ble) $\ell(A) < \infty$, then $\#(X \cap A) \sim \text{Poi}(c\,\ell(A))$ for some $c > 0$ fixed.

ii) For $A, B \subseteq \mathbb{R}^d$, both Borel m'ble with $\ell(A), \ell(B) < \infty$. If A and B are disjoint, then $\#(X \cap A) \perp\!\!\!\perp \#(X \cap B)$.

eg. For a poisson point process on $\mathbb{R}_{\geq 0}$, we get the usual poisson process with $\text{Exp}(c)$ arrival times.

(iii) Consequence: Given $A \subseteq \mathbb{R}^d$ with $\ell(A) < \infty$, given that

$$\#(X \cap A) = n$$

these n points are uniformly distributed on A, i.e., $\forall B \subseteq A$ m'ble,

$$\{\#(X \cap B) \mid \#(X \cap A) = n\} \sim \text{Unif}(\quad) \text{ with mean } n\frac{\ell(B)}{\ell(A)}.$$

Any two of (i), (ii) and (iii) imply the third. We thus get an equivalent defn of the Poisson Process.

- This is called a Poisson Point process with intensity c (or λ).

- We have a iid positive sequence of positive $\mathbb{R}$-valued r.v.'s $\rho_1, \rho_2, \dots$ (indep of (X, λ)) and we look at balls $S_1, S_2, \dots$ where $S_i = B(\underline{0}, \rho_i)$ in $\mathbb{R}^d$.

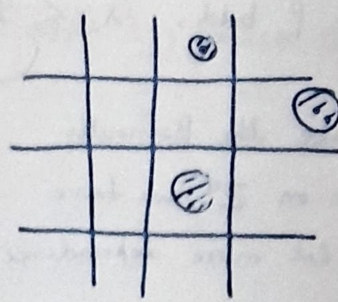
- For the process (X, λ), we split up $\mathbb{R}^d$ into boxes of the form

$$[a_1, a_1+1) \times \dots \times [a_d, a_d+1)$$

Each of these boxes have finitely many pts of X, and thus $\# X$ is countable a.s. Thus, we can label the points of X by an prescription.

We define the covered region as

$$C := \bigcup_{i=1}^{\infty} (\underset{\sim}{x_i} + S_i)$$



$\mathbb{R}^d \setminus C \longrightarrow$ Vacant region.

This triplet (X, λ, ρ) is called the Boolean model, and it is on this model that we will define Continuum Percolation.

- Let $X \sim PPP(\lambda)$ on $\mathbb{R}^d$ and indep $Y \sim PPP(\mu)$ on $\mathbb{R}^d$, then $X + Y \sim PPP(\lambda+\mu)$.

- Poisson process under affine transformations:

  For $M : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $\det M = 1$, the process

  $$MX := \{Mx \mid x \in X\} \quad \text{on} \quad \{Mz \mid z \in \mathbb{R}^d\}$$

  is also a $PPP(\lambda)$ process.

- By a suitable coupling argument: $\hat{C}(\lambda) \subseteq \hat{C}(\lambda+\mu)$ [Use superposition idea]

Let C_0 be the connected component ~~cc~~ of C containing 0.
We henceforth ~~do~~ fix the process $\underset{\sim}{\rho}$, and Parametrize by λ.

$$\lambda_c = \inf\{\lambda : \mathbb{P}_\lambda(\mathrm{diam}(C_0) = \infty) > 0\}$$
$$= \inf\{\lambda : \mathbb{P}_\lambda(\ell(C_0) = +\infty) > 0\}$$
$$= \inf\{\lambda : \mathbb{P}_\lambda(\#(X \cap C_0) = \infty) > 0\}$$

We can also talk about the percolation ~~of~~ on V. $V_0 =$ cc of 0 in V.

$$\lambda_c^* = \sup\{\lambda : \mathbb{P}_\lambda(\mathrm{diam}(V_0) = \infty) > 0\}$$
$$= \sup\{\lambda : \mathbb{P}_\lambda(\ell(V_0) = \infty) > 0\}$$

- For ρ a bounded r.v., we indeed have $\lambda_c = \lambda_c^*$ in 2 dimensions.

  This is analogous to $p_c = \frac{1}{2}$ ~~ab~~ on $\mathbb{Z}^2$.

- For ρ bdd, $\lambda_c < \lambda_c^*$ in higher dimensions. (PhD. — Anish Sarkar).
  → There is a region in which an inf. Vac comp & inf cov. comp coexist.

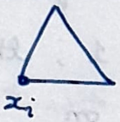
Unlike the Bernoulli perc on $\mathbb{Z}^d$, we have a lot more dependence here.

- We can generalize S_i's to other shapes of area 1.

- P_k — regular polygon of k-sides with area 1. Taking these shapes, we can talk about $\lambda_c^{(k)}$ and $\lambda_c^{(\infty)}$. How do these compare?

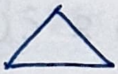
<u>Q</u>: Where is x_i in $(x_i + \Delta)$?

OR



→ Either of these will give us the same answer / model
→ Follows from affine Invariance.

→ For the same reason, the orientation of the Δ does not matter. Neither does the precise shape matter



<u>OR</u>

⇝ equivalent. [Give a proof of this]

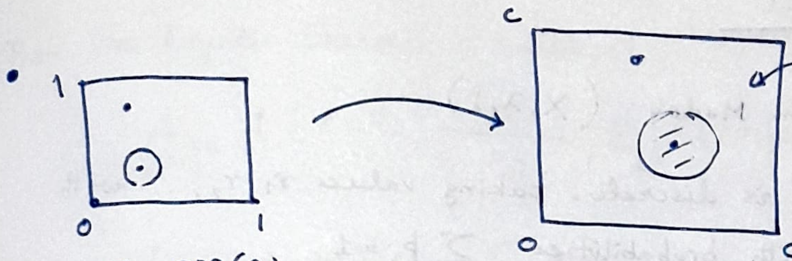
<u>Result</u>: $\lambda_c^{(k)} < \lambda_c^{(k+1)}$, i.e., it is harder to percolate on larger sided polygons.

<u>COVERED VOLUME FRACTION</u>: Consider the unit square $[0,1)^d$ in $\mathbb{Z}^d$ and consider $\mathbb{E}_\lambda\left(\ell\left(C \cap [0,1)^d\right)\right)$ → Covered volume fraction.

- $L_n := \ell\left(C \cap [0,n)^d\right)$

Then, $\frac{L_n}{n^d} \xrightarrow{a.s.} \mathbb{E}_\lambda\left(\ell\left(C \cap [0,1)^d\right)\right) = \mathrm{CVF}(\lambda)$

This follows from SLLN, by breaking $[0,n)^d$ into n^d many blocks (iid).



PPP$\left(\frac{\lambda}{c^d}\right)$ process now.
$\overset{\|}{\mu}$

$X \sim PPP(\lambda)$

$\therefore$ If $\lambda > \lambda_c(\rho)$, then $\mu > \lambda_c(c\rho)$ [$\because$ the percolation property does not change]

$\therefore$ $\lambda_c(1) = \lambda_c(r)\cdot r^d$ immediately, from the scaling properties.

Let S = Ball of a fixed radius r, i.e., $S = B(0,r)$.

$$\therefore \mathbb{P}(\underset{\sim}{0} \in V) = \mathbb{P}(C \cap B(0,r) = \phi)$$
$$= e^{-\lambda \operatorname{Vol}(B(0,r))} = e^{-\lambda \pi_d r^d}$$
$$\therefore \mathbb{P}(\underset{\sim}{0} \in C) = e^{-\lambda \pi_d r^d}.$$

Suppose now we have $S_1 = \begin{cases} B(0,r) \text{ wp } \frac{1}{2} \\ B(0,s) \text{ wp } \frac{1}{2} \end{cases}$

At each point, we toss a coin and get the ball

Superpose a PPP$\left(\frac{\lambda}{2}\right)$ with $B(0,r)$ and a PPP$\left(\frac{\lambda}{2}\right)$ with $B(0,s)$.

<u>H.W.</u> $$\mathbb{P}_\lambda(\underset{\sim}{0} \in C) = 1 - e^{-\lambda \pi_d \mathbb{E}(\rho^d)}$$

In particular, $\mathbb{P}_{\lambda_c}(\underset{\sim}{0} \in C) = 1 - e^{-\lambda_c \pi_d \mathbb{E}(\rho^d)}$

Fix ρ s.t. $\operatorname{Vol}(B(0,\rho)) = r$. Then,

$$\mathbb{P}_{\lambda_c(\rho)}(0 \in C) = 1 - e^{-\lambda_c \pi_d \mathbb{E}(\rho^d)} = 1 - e^{-\lambda_c(1)} \quad [\lambda_c(1) = \lambda_c(r)\, r^d]$$
$$= c\,CVF(1)$$

$\hookrightarrow$ critical covered vol fraction.

There was a conjecture in physics:

$$\mathbb{P}_{\lambda_c}(0 \in C) = c\,CVF(1) \text{ for any } \rho.$$

[We have shown that this holds for ρ fixed, ~~and ρ random when the shapes are balls~~.]

Lecture - 20

We have the Poisson Boolean Model (X, λ, ρ).

Suppose random variable ρ is discrete, taking values $r_1, r_2, \dots$ with probabilities $p_1, p_2, \dots$, with probabilities $\sum p_i = 1$

Then we can decompose (X, λ, ρ) into a collection $\{(X_i, \lambda_i, r_i) : i \geq 1\}$ of indep. Poisson Boolean Models. If $\lambda_i = \lambda p_i$, then the superposition of these models will give us (X, λ, ρ).

Let $\pi_d = \ell(B_d(0,1))$.

$$\therefore \mathbb{P}_\lambda(\underset{\sim}{0} \in V) = \prod_{i=1}^{\infty} \mathbb{P}_{\lambda_i}(\text{No point of } X_i \text{ lies in a radius } r_i \text{ from the origin})$$

$$= \prod_{i=1}^{\infty} e^{-\lambda p_i \pi_d r_i^d} = e^{-\lambda \pi_d \sum p_i r_i^d}$$

$$= e^{-\lambda \pi_d \mathbb{E}\rho^d}.$$

$\therefore$ For a discrete r.v., we have obtained

$$\mathbb{P}_\lambda(\underset{\sim}{0} \in V) = e^{-\lambda \pi_d \mathbb{E}\rho^d}.$$

For a general r.v. $\rho > 0$, we first truncate and then approximate by discrete r.v.. Taking limits, we get

$$\left.\mathbb{P}_\lambda(\underset{\sim}{0} \in V) = e^{-\lambda \pi_d \mathbb{E}\rho^d}\right\} \quad \text{[H.W.]}$$

<u>Conclusion:</u> By translation invariance,

i) $\ell(V) = 0$ if $\mathbb{E}\rho^d = \infty$

ii) Since the model is Ergodic and shapes are convex,

$$V = \emptyset \;\; \underline{\text{iff}} \;\; \mathbb{E}\rho^d = \infty.$$

- The quantity $e^{-\lambda \pi_d \mathbb{E}\rho^d}$ has another interpretation. Let $D_n = [-n, n]^d$ and consider $\ell(C \cap D_n)$. This is the sum of the Lebesgue measures of the ~~covered regions~~ cov reg. of $(2n)^d$ unit cubes comprising this.

Then the Ergodic Theorem is valid, i.e.,

$$\frac{1}{(2n)^d}\, \ell(C \cap D_n) \xrightarrow[L^1]{a.s.} \mathbb{E}_\lambda\left[\ell(C \cap [0,1]^d)\right]$$

But $$\mathbb{E}(\ell(C\cap[0,1]^d)) = \int_{[0,1]^d} \mathbb{P}_\lambda(\underset{\sim}{x} \in C)\, d\underset{\sim}{x} = \int_{[0,1]^d} \mathbb{P}_\lambda(\underset{\sim}{0} \in C)\, d\underset{\sim}{x}$$

$$= \left(\int_{[0,1]^d} d\underset{\sim}{x}\right)\cdot\left(1 - \mathbb{P}_\lambda(\underset{\sim}{0} \in V)\right) = 1 - e^{-\lambda \pi_d \mathbb{E}\rho^d}.$$

This is called the covered volume fraction ;

$$CVF(X,\lambda,\rho) = 1 - e^{-\lambda\pi_d \mathbb{E}\rho^d}.$$

Defn: The critical covered volume fraction (cCVF) of a Poisson Boolean Model is

$$A_c(\rho) = 1 - e^{-\lambda_c(\rho)\,\pi_d\,\mathbb{E}\rho^d}$$

- We had seen earlier that $\lambda_c(r)\, r^d = \lambda_c(1)$ for any $r>0$. Thus, if ρ is a constant $\rho \equiv r$, then

$$A_c(\rho) = 1 - e^{-\lambda_c(r) r^d \pi_d} = 1 - e^{-\lambda_c(1)\pi_d} = A_c(1) = A_c.$$

In particular, $A_c(\rho)$ is a constant.

In Physics, it was believed that A_c was the parameter for Poisson Boolean model. But this was false, as will be proven below.

Thm: For a Poisson Boolean Model (X,λ,ρ) in $\mathbb{R}^d$, $\exists$ a r.v. ρ taking values a, b $(a \neq b)$ wp $1-p$ and p respectively, with $A_c(\rho) > A_c$.

TOOLS :

- FKG-inequality:

We have (X,λ,ρ) on some prob. space $(\Omega, \mathcal{F}, \mathbb{P})$.

- For $\omega, \omega' \in \Omega$, we say $\omega \preceq \omega'$ if

i) $X(\omega) \subseteq X(\omega')$ ii) For $x \in X(\omega)$, $\rho_x(\omega) = \rho_x(\omega')$

Def$^{\underline{n}}$: An event $A \in \mathcal{F}$ is said to be increasing if

$$\mathbb{1}_A(\omega) \leq \mathbb{1}_A(\omega') \quad \text{for } \omega \lesssim \omega'$$

It is decreasing if $\mathbb{1}_A(\omega) \geq \mathbb{1}_A(\omega')$ for $\omega \lesssim \omega'$.

Thm: (FKG Inequality) For $A, B \in \mathcal{F}$, either both increasing or both decreasing, we have $\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\,\mathbb{P}(B)$.

Proof: Consider the lattice

$$\mathbb{L}_n = \left(\frac{1}{2^n}\mathbb{Z}^d\right)^d \times \frac{1}{2^n}\mathbb{N}$$

For $\underset{\sim}{k} = (k_1, \dots, k_d) \in \mathbb{Z}^d$ and $s \in \mathbb{N}$, let $C_{\underset{\sim}{k}, s}$ denote the cell

$$C_{\underset{\sim}{k}, s} = \left\{ (\underset{\sim}{x}, r) : \frac{k_i - 1}{2^n} < x_i \leq \frac{k_i}{2^n} \ \forall i, \ \frac{s-1}{2^n} < r \leq \frac{s}{2^n} \right\}$$

$\longrightarrow$ 1 cell in $\mathbb{L}_n$.

Cell in $\mathbb{L}_n$

Evidently, these cells give a disjoint partition of $\mathbb{L}_n$.

In particular,
$$\bigcup_{\substack{\underset{\sim}{k} \in \mathbb{Z}^d \\ s \in \mathbb{N}}} = \mathbb{R}^d \times \mathbb{R}_+ \longrightarrow \text{disjoint partition.}$$

Given $C = C_{k,s}$, let $N_n(C) = \#$ of Poisson points of (X, λ, ρ) which fall in C and whose assoc. balls have radius $\left(\frac{s-1}{2^n}, \frac{s}{2^n}\right]$.

Let $\mathcal{F}_n \subseteq \mathcal{F}$ be the σ-algebra generated by $\{N_n(C) : C \text{ is a cell in } \mathbb{L}_n\}$.

For an event $A \in \mathcal{F}$, $\{\mathbb{E}(\mathbb{1}_A \mid \mathcal{F}_n) : n \geq 1\}$ is a positive martingale w.r.t. $\{\mathcal{F}_n\}_{n \geq 1}$. Then by Martingale Conv. thm,

$$\mathbb{E}(\mathbb{1}_A \mid \mathcal{F}_n) \xrightarrow{n \to \infty} \mathbb{1}_A$$

Now, the collection $\{N_n(C) : C \in \mathbb{L}_n\}$ is a collection of independent random variables. Also, $\mathbb{E}(\mathbb{1}_A \mid \mathcal{F}_n)(\omega)$ is a f$^{\underline{n}}$ of $\{N_n(C) \mid C \in \mathbb{L}_n\}$.

So, if A is an increasing event, $\mathbb{E}(\mathbb{1}_A \mid \mathcal{F}_n)$ is an increasing function.

Note that we have defined everything on a Discrete setup. Thus, we can use our earlier Version of FKG inequality. Thus,

$$\mathbb{E}\left(\mathbb{E}(1_A|\mathcal{F}_n)\,\mathbb{E}(1_B|\mathcal{F}_n)\right) \geq \mathbb{E}\left(\mathbb{E}(1_A|\mathcal{F})\right)\mathbb{E}\left(\mathbb{E}(1_B|\mathcal{F})\right)$$
$$= \mathbb{E}1_A\,\mathbb{E}1_B$$

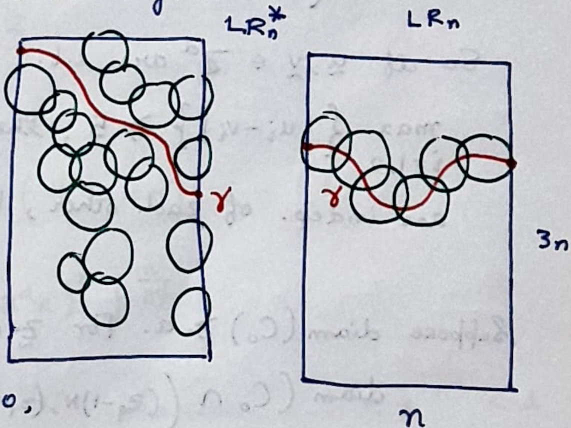
By DCT, as $\mathbb{E}(1_A|\mathcal{F}_n) \longrightarrow 1_A$ and $\mathbb{E}(1_B|\mathcal{F}_n) \longrightarrow 1_B$, we have

$$\mathbb{E}(1_A 1_B) \geq \mathbb{E}1_A\,\mathbb{E}1_B$$

$$\Longrightarrow \mathbb{P}(A\cap B) \geq \mathbb{P}(A)\cdot\mathbb{P}(B). \quad \blacksquare$$

<u>Exponential Decay:</u>

<u>Def$^{\underline{n}}$:</u> Let LR_n denote the event that the occupied region C admits an occupied path γ connecting two opposite faces in the shorter direction of the rectangular box $[0,n]\times[0,3n]\times\ldots\times[0,3n]$ in $\mathbb{R}^d$. LR_n^* is defined similarly for the vacant region.



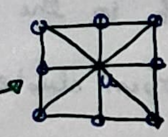
<u>Prop$^{\underline{n}}$:</u> Let (X,λ,ρ) be a Poisson Boolean model with $\rho \leq R$. Then $K_0 \in (0,1)$ $\left[K_0 = (e3^d)^{-11^d-1}\right]$ s.t. if $\mathbb{P}_\lambda(LR_N) < K_0$ for some $N > R$, then for all sufficiently large $a > 0$, we have

$$\mathbb{P}_\lambda(\operatorname{diam}(C_0) \geq \alpha) \leq K_1 e^{-K_2 a}$$

for constants $K_1, K_2 > 0$ depending only on R.

<u>Proof:</u> $(d=2)$ Let $\underset{\sim}{u} = (u_1,v_2)$ and $\underset{\sim}{v} = (v_1,v_2)$ in $\mathbb{Z}^d$. We say $\underset{\sim}{u}$ & $\underset{\sim}{v}$ are adjacent if $\|\underset{\sim}{u}-\underset{\sim}{v}\|_\infty \leq 1$, i.e.,

$$\max_i \{|u_i - v_i|\} \leq 1$$



adj. vertices.

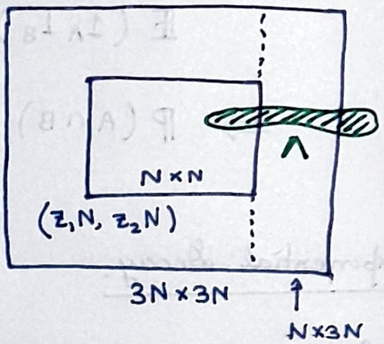
Effective lattice.

We now do site percolation on this lattice.

A vertex $\underset{\sim}{z} \in \mathbb{Z}^2$ is open if $\exists$ a conn. comp in C, say Λ, of the PB model s.t.

i) $\Lambda \cap \left((z_1 N, (z_1+1)N] \times (z_2 N, (z_2+1)N] \right) \neq \emptyset$

ii) $\Lambda \cap \left(((z_1-1)N, (z_1+2)N] \times ((z_2-1)N, (z_2+2)N] \right)^c \neq \emptyset$



$$\therefore\ p = \mathbb{P}_\lambda(\underset{\sim}{z} \text{ is open})$$
$$= 2 \sum_{i=1}^{d} \mathbb{P}_\lambda(LR_N)$$
$$\leq 2d K_0 \text{ by our assumption}$$

Now, as each ball is of radius at most R, the point $\underset{\sim}{z}$ being open in $\mathbb{Z}^2$ depends only on the config. in the rectangle

$$((z_1-1)N - R, (z_1+2)N+R] \times ((z_2-1)N - R, (z_2+2)N+R]$$

So if $\underset{\sim}{u}, \underset{\sim}{v} \in \mathbb{Z}^d$ are s.t.

$\max_{i=1,2} \{ |u_i - v_i| \} \geq 5$, then the open/closedness of u, v are indep. of each other, ~~taking N > a large enough~~ as $N > R$.

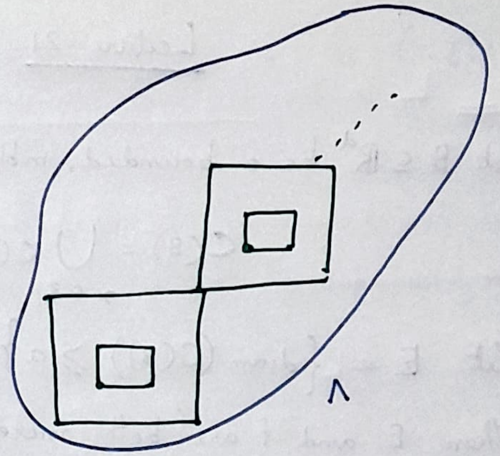
Suppose $\operatorname{diam}(C_0) \geq a$. For $\underset{\sim}{z} \in \mathbb{Z}^2$; ~~$\operatorname{diam}(C_0)$~~

$$\operatorname{diam}\left(C_0 \cap \left(((z_1-1)N, (z_1+2)N] \times ((z_2-1)N, (z_2+2)N] \right)\right) \leq 3N\sqrt{2}$$

So, if $\operatorname{diam}(C_0) \geq a$, $\exists$ at least $\frac{a}{N\sqrt{2}}$ vertices which satisfy (i) of being open and if $a > \frac{3N}{\sqrt{2}}$, then each of these $\frac{a}{N\sqrt{2}}$ many vertices satisfying (ii) of being open.

If in the renormalized lattice $N\mathbb{Z} \times N\mathbb{Z}$, C' denotes the open cluster of the origin in site percolation, then diam $C_0 \geq a > 3N\sqrt{2}$

$\Rightarrow \# C' \geq \frac{a}{N\sqrt{2}}$



Given set S_n of n vertices of $\mathbb{Z}^d$, since the states of 2 vertices are indep whenever they are at a distance 5 (in l_∞ sense), then the set S_n has at least $\frac{n}{(11)^d}$ vertices whose configurations are indep of each other.

$\therefore\ \mathbb{P}_\lambda$(all vertices in S_n are open) $\leq p^{\frac{n}{(11)^d}}$.

Thus, $\mathbb{P}_\lambda(\# C' = n) = \sum_{S_n} \mathbb{P}_\lambda(C' = S_n)$, where the sum is over all connected sets S_n of n-vertices of $\mathbb{Z}^2$, containing 0.

Let b_n = # such sets. Then $b_n \leq (e3^d)^n$. [Try yourself]

$\therefore\ \mathbb{P}_\lambda(\# C' = n) \leq (3^d e)^n p^{\frac{n}{11^d}}$

Combining everything,

$$\mathbb{P}_\lambda(\text{diam}(C_0) \geq a) \leq \sum_{n \geq \frac{a}{N\sqrt{2}}} \mathbb{P}_\lambda(\# C' = n)$$

$$\leq \sum_{n \geq \frac{a}{N\sqrt{2}}} (3^d e)^n p^{\frac{n}{11^d}}$$

$\leq K_1 e^{-K_2 a}$ whenever $3^d e\, p^{\frac{1}{11^d}} < 1$.

Lecture - 21

1) Let $B \subseteq \mathbb{R}^d$ be a bounded, mble set, $0 \in B$. Define

$$C(B) = \bigcup_{x \in B} C(x)$$

Let $E := \{\text{diam}(C(B)) \geq a\}$ and $F = \{B \subseteq C\}$

Then E and F are both increasing events and thus

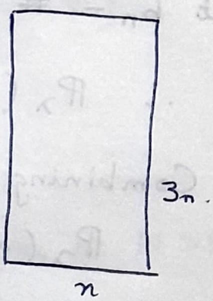
$$\mathbb{P}_\lambda(E \cap F) \geq \mathbb{P}_\lambda(E)\, \mathbb{P}_\lambda(F)$$

Note that as B is bounded, $\mathbb{P}_\lambda(F) > 0$. We call this probability $K(\lambda, B) = \mathbb{P}_\lambda(F)$. Therefore

$$\mathbb{P}_\lambda(E \cap F) \geq \mathbb{P}(E)\,\mathbb{P}(F) \geq K(\lambda, B)\, \mathbb{P}(\text{diam } C(B) \geq a)$$

$$\therefore \mathbb{P}_\lambda(\text{diam } C(0) \geq a) \geq K(\lambda, B)\, \mathbb{P}(\text{diam } C(B) \geq a) \longrightarrow (*)$$

Last time, we looked at the event LR_n. Unfortunately this is not monotonically increasing or decreasing with n.



Define

$$\lambda_s(\rho) = \inf\{\lambda : \limsup \mathbb{P}_\lambda(LR_n) > 0\}$$

<u>Thm</u>: For $\rho \leq R$ for some $R > 0$, we have

$$\lambda_s(\rho) = \lambda_c(\rho)$$

[We assume this for now]

<u>Proof of the main theorem</u>: Let $0 < r_1 < r_2 < \infty$ be arbitrary. Recall that
<u>(d=2)</u> the critical covered volume fraction

$$A_c = 1 - e^{-\lambda_c(1)\pi_d}$$

Fix $\varepsilon, \delta > 0$ s.t. $(2-\varepsilon-\delta)A_c - (1-\varepsilon)(1-\delta)A_c^2 > A_c$

Now choose $\lambda_2 < \lambda_c(r_2)$ s.t. the CVF of (X, λ_2, r_2) is $(1-\varepsilon)A_c$.

This can be done as the CVF of (X, λ, r_2) is cont. in λ.

Also, choose $\lambda_1 < \lambda_c(r_1)$ s.t. the CVF of (X, λ_1, r_1) is $(1-\delta)A_c$.

So, each of these processes are subcritical.

Now, $\mathbb{P}(\underset{\sim}{0} \in C_1$ or $\underset{\sim}{0} \in C_2) = \mathbb{P}(\underset{\sim}{0} \in \mathcal{C}$, $\mathcal{C}$ - Covering of the superposed model)

$$= (1-\varepsilon)A_c + (1-\delta)A_c - (1-\varepsilon)(1-\delta)A_c^2 > A_c$$

Consider the model (X_1, λ_1, r_1) and take $\alpha < 1$.

We now scale the setup by α. Then the new radius becomes αr_1 and intensity becomes $\frac{\lambda_1}{\alpha^d}$.

$$\therefore \quad 1 - e^{-\lambda_1 r_1^d \pi_d} = 1 - e^{-\frac{\lambda_1}{\alpha^d}(\alpha r_1)^d \pi_d}$$

$$\therefore \text{ CVF of } \left(X, \frac{\lambda_1}{\alpha^d}, \alpha r_1\right) = \text{CVF}(X, \lambda_1, r_1) = (1-\delta)A_c$$

$\therefore$ CVF of the superposition of $(X, \frac{\lambda_1}{\alpha^2}, \alpha r_1)$ and (X, λ_2, r_2) is still strictly bigger than A_c.

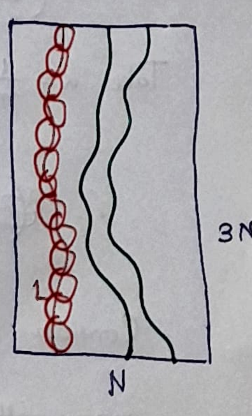
<u>Claim</u>: $\exists\ 0 < \alpha < 1$ s.t. this superposed model is subcritical.

[Doing this shows that CVF is not the parameter responsible for criticality]

i.e., A_c (Superposed model at criticality) $> A_c$

<u>Proof</u>: Fix $0 < K < K_0$. As $\lambda_2 < \lambda_c(r_2) = \lambda_S(r_2)$. In particular, $\exists N$ s.t. $\mathbb{P}_{\lambda_2}(LR_N) < \frac{1}{3}K$

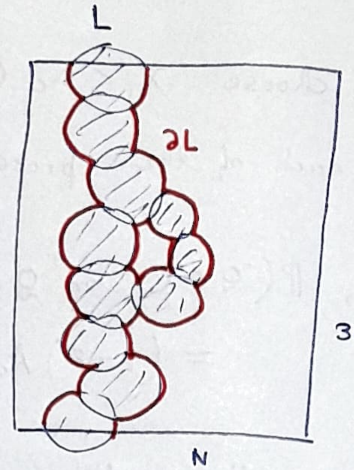
Consider the $N \times 3N$ rectangle in figure. If $\nexists$ any occupied L-R crossing of $[0,N] \times [0,3N]$, then $\exists$ a vacant T-B crossing of this rectangle.



Let L be the leftmost T-B crossing of the box. In the box $[-R, N+R] \times [-R, 3N+R]$, $\exists$ finitely many Poisson points a.s.

T-B crossing

Note: L is the leftmost region, not path.



Then the boundary ∂L of L has only finitely many components. So, for large n,

$$E_n := \left\{ \begin{array}{l} L \text{ exists and all conn comp. of} \\ \partial L \cap \left(\text{Int}\left([0,N] \times [0,3N]\right)\right) \\ \text{are separated by a distance of} \\ \text{at least } \frac{1}{n} \text{ from each other} \end{array} \right\}$$

≡ [There is good overlap between all balls] OR [The balls do not $\cap$ and are far from each other.]

As $n \uparrow \infty$, $E_n \uparrow$. Again, get n_0 s.t.

$$\mathbb{P}_{\lambda_2}(E_{n_0}) > 1 - \tfrac{1}{2}K$$

This follows as $\mathbb{P}_{\lambda_2}(E_n \mid L \text{ exists}) \longrightarrow 1$ ~~and~~ as $n \uparrow \infty$, and

$$\mathbb{P}_{\lambda_2}(L \text{ exists}) > 1 - \tfrac{1}{3}K \longrightarrow \text{(A)}$$

Now, let $B_s = [-s, s]^2$. Since $\lambda_1 < \lambda_c(r_1)$, thus

$$\mathbb{P}_{\lambda_1}(\text{diam } C(0) \geq a) \text{ decays exponentially}$$

$\therefore$ Using ⊛, $\mathbb{P}_{(\lambda, r_1)}(\text{diam } C(B_1) \geq b) \leq C_3 e^{-C_4 b}$ for all $b > R$

and constants $C_3, C_4 > 0$.

Now, scaling by a factor $\alpha < 1$ gives

$$\mathbb{P}_{\left(\frac{\lambda_1}{\alpha^2}, \alpha r_1\right)}\left(\text{diam } C(B_\alpha) \geq \alpha b\right) \leq C_3 e^{-C_4 b}$$

Take $\alpha = \frac{1}{m}$ for some large $m \in \mathbb{N}$ and take $b = \frac{1}{2\alpha n_0}$ where n_0 as in (A)

$\longrightarrow$ (B)

$$\therefore \mathbb{P}_{(m^2\lambda_1, \frac{1}{m} r_1)}\left(\text{diam } C(B_{\frac{1}{m}}) \geq \frac{1}{2n_0}\right) \leq C_3 e^{-C_4 \cdot \frac{m}{2n_0}}$$

Consider $\left(\frac{1}{m}\mathbb{Z}\right)^2 \cap \left([0,N] \times [0,3N]\right)$. This has $3N^2m^2 < \infty$ many cells.

Let $b_1, b_2, \dots, b_{3N^2m^2}$ be a labelling of those cells, and consider the ~~event that the cluster~~ following event:

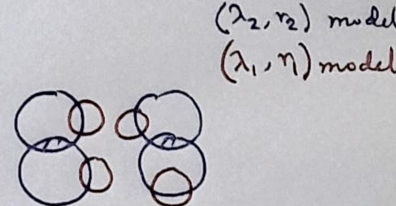
$$F_{n_0}^{m} = \bigcup_{i=1}^{3N^2m^2} \left\{ \operatorname{diam} C(b_i) \geq \frac{1}{2n_0} \right\}$$

Using Ⓑ and translation invariance followed by union bound, we have

$$\mathbb{P}_{(m^2\lambda_1, \frac{r_1}{m})} \left(F_{n_0}^{m} \right) \leq 3N^2m^2 \, c_3 e^{-c_4 \frac{m}{2n_0}} \longrightarrow 0 \text{ as } m \longrightarrow \infty$$

In other words, $\exists\, m_0$ s.t.

$$\mathbb{P}_{(m_0^2\lambda_1, \frac{r_1}{m_0})} \left(F_{n_0}^{m_0} \right) < \frac{1}{3}K$$



Now, to bridge the gaps from the (λ_2, r_2) model in $[0,N]\times[0,3N]$ by those in the scaled (λ_1, r_1) balls, we need $\operatorname{diam}(b_i) > \frac{1}{2n_0}$ for some b_i.

The superposition of (X, λ_2, r_2) and $(X, m_0^2\lambda_1, \frac{r_1}{m_0})$ is the Poisson Boolean model $(X, m_0^2\lambda_1 + \lambda_2, \rho)$ where

$$\rho = \begin{cases} r_2 & \text{wp } \dfrac{\lambda_2}{m_0^2\lambda_1 + \lambda_2} \\[2ex] \dfrac{r_1}{m_0} & \text{wp } \dfrac{m_0^2\lambda_1}{m_0^2\lambda_1 + \lambda_2} \end{cases}$$

Then the probability of LR_N is this model is $\frac{1}{2}K + \frac{1}{3}K < K < K_0$.

So, by the exponential decay proposition, this model is subcritical. ▨