Lecture - 15

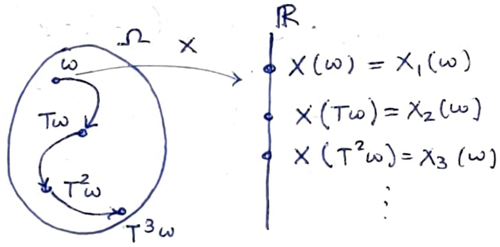
1/10/24

- **Def^n**: A process $X_1, X_2, \ldots$ is called stationary if for every $k \geq 1$
$$(X_1, X_2, \ldots) \stackrel{d}{=} (X_{k+1}, X_{k+2}, \ldots)$$
In other words, for every $x_1, \ldots, x_n$, $n \geq 1$
$$\mathbb{P}(X_1 \leq x_1, \ldots, X_n \leq x_n) = \mathbb{P}(X_{k+1} \leq x_1, \ldots, X_{k+n} \leq x_n)$$

- **Def^n**: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a prob space and $T: \Omega \to \Omega$ a mble mapping. T is said to be a measure preserving transformation (mpt) if
$$\mathbb{P}(T^{-1}(A)) = \mathbb{P}(A) \quad \forall A \in \mathcal{F}.$$

(on $(\Omega, \mathcal{F}, \mathbb{P})$ underlying space)

Prop^n: Let T be a mpt and X a random variable. The sequence $X = X_1, X_2, \ldots$ given by $X_n(\omega) = X(T^{n-1}\omega)$ for every $n \geq 1$ is a stationary ~~process~~ sequence of random variables.



Proof: 1) Show that X_n is a rv.
(composition of m'ble f^{n}s are m'ble)

2) Show that for $a_1, \ldots, a_n \in \mathbb{R}$ and
$$A_n := \{X_1 \leq a_1, \ldots, X_n \leq a_n\}$$
$$= \{\omega \mid X(\omega) \leq a_1, X(T(\omega)) \leq a_2, \ldots, X(T^{n-1}(\omega)) \leq a_n\}$$
$$B_n = \{X_2 \leq a_1, \ldots, X_{n+1} \leq a_n\}$$
$$= \{\omega \mid X(T\omega) \leq a_1, \ldots, X(T^n\omega) \leq a_n\}.$$
So $\omega \in B_n$ iff $T\omega \in A_n$, i.e., $B_n = T^{-1}A_n$
$$\therefore \mathbb{P}(B_n) = \mathbb{P}(T^{-1}A_n) = \mathbb{P}(A_n).$$
Using this, we can prove that (X_n) is a stationary (by Induction). ☒

Def^n:
Let $X_1, X_2, \ldots$ be a stationary process on $(\Omega, \mathcal{F}, \mathbb{P})$. The co-ord. rep^n process is a process on $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), \hat{\mathbb{P}})$, say $\hat{X}_1, \hat{X}_2, \ldots$, s.t.
for $\underline{x} = (x_1, x_2, \ldots) \in \mathbb{R}^\infty$, $\hat{X}_n(\underline{x}) = x_n$ and
$$\hat{\mathbb{P}}(\hat{X}_1 \leq x_1, \ldots, \hat{X}_n \leq x_n) = \mathbb{P}(X_1 \leq x_1, \ldots, X_n \leq x_n)$$
$$\parallel$$
$$\hat{\mathbb{P}}((-\infty, x_1] \times \cdots \times (-\infty, x_n])$$

Such a $\hat{\mathbb{P}}$ on $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$ for any stochastic process $\mathbb{P}$ on $(\Omega, \mathcal{F})$.

Def^n: Let $S : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ be defined as $S(x_1, x_2, \dots) = (x_2, x_3, \dots)$.
Then S is called shift operator.

Prop^n: Let $X_1, X_2, \dots$ be a stationary process. Then S is a measure preserving transformation on $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), \hat{\mathbb{P}})$

Proof: 1) S is mble $\to [\, S^{-1}(A_n) \in \mathcal{B}(\mathbb{R}^\infty)$ always $]$

2) Take $A_n = \{ \underset{\sim}{x} \mid x_1 \le a_1, \dots, x_n \le a_n \}$

$B_n = \{ \underset{\sim}{x} \mid x_2 \le a_1, \dots, x_{n+1} \le a_n \}$

$\therefore \quad B_n = S^{-1} A_n$, and

$$\begin{aligned}\hat{\mathbb{P}}(B_n) &= \hat{\mathbb{P}}(\hat{X}_2 \le a_1, \dots, \hat{X}_{n+1} \le a_n) \\ &= \mathbb{P}(X_2 \le a_1, \dots, X_{n+1} \le a_n) \\ &= \mathbb{P}(X_1 \le a_1, \dots, X_n \le a_n) = \hat{\mathbb{P}}(\hat{X}_1 \le a_1, \dots, \hat{X}_n \le a_n) \\ &= \hat{\mathbb{P}}(A_n) \quad \Rightarrow S \text{ is a mpt}\end{aligned}$$

§ Ergodicity: Let T be a mpt on $(\Omega, \mathcal{F}, \mathbb{P})$.

Def^n: 1) A set $A \in \mathcal{F}$ is invariant if $A = T^{-1}A$

2) A set $A \in \mathcal{F}$ is a.s. invariant if $\mathbb{P}(A \,\Delta\, T^{-1}A) = 0$.

(**) HW: 1) Show that $\mathcal{I} = \{ A \mid T^{-1}A = A \}$ is a σ algebra, called the invariant σ-algebra.

2) Show that the collection of almost surely invar. sets is just a completion of $\mathcal{I}$ in Ω.

Def^n: A mpt on $(\Omega, \mathcal{F}, \mathbb{P})$ is ergodic if $\mathbb{P}(A) = 0$ or 1 for every $A \in \mathcal{I}$.

Def^n: Let X be a r.v. and T a mpt on $(\Omega, \mathcal{F}, \mathbb{P})$. The random variable X is T-invariant if $X(\omega) = X(T\omega)$.

Prop^n: X is invariant iff X is $\mathcal{I}$'mble.

Proof: Suppose X is invariant. Then

$A = \{ \omega : X(\omega) \le x \} = \{ \omega : X(T\omega) \le x \} = T^{-1}A$

$\Rightarrow X$ is $\mathcal{I}$ mble.

Conversely, let $\mathcal{L}$ be the class of all $\mathcal{I}$ m'ble random variables.

Then,
1) If $A \in \mathcal{I}$, then $1_A \in \mathcal{L}$
2) It is closed under finite linear combinations
3) If $X_n \in \mathcal{L}$ and $X_n \uparrow X$, then $X \in \mathcal{L}$

So $\mathcal{L}$ contains all non-negative rv's on $(\Omega, \mathcal{I})$

4) Since every r.v. can be written as $X = X^+ - X^-$, $\mathcal{L} \supseteq \mathcal{I}$ mble fns.

<u>Propn</u>: Let T be a mpt on $(\Omega, \mathcal{F}, \mathbb{P})$. Then T is ergodic iff every invariant r.v. is a constant a.s.

We have used this for # conn. comp. for the Percolation process.

<u>Proof</u>: Suppose every inv. rv is a constant a.s. Then, for $A \in \mathcal{I}$, 1_A is a constant a.s., i.e., $\mathbb{P}(A) = 0$ or 1.

Conversely $\{w : X(w) \leq a\} \in \mathcal{I} \Rightarrow \mathbb{P}(\overset{w:}{X(w)} \leq a) = 0$ or 1, for any invariant r.v. X. So, $X = a_0$ where $a_0 = \underbrace{\inf}_{\text{(actually a min here)}} \{a \mid \mathbb{P}(X(w) \leq a) = 1\}$

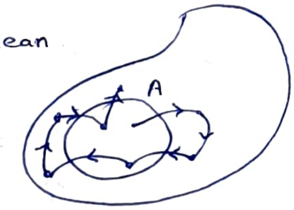
<u>Ergodic Thm</u>: Let T be a mpt on $(\Omega, \mathcal{F}, \mathbb{P})$. For any random variable X with $\mathbb{E}|X| < \infty$, we have, as $n \to \infty$,

$$\frac{1}{n} \sum_{k=0}^{n} X(T^k w) \longrightarrow \mathbb{E}(X|\mathcal{I}) \quad \text{a.s. and in mean.}$$

In addition, if T is ergodic, then

$$\frac{1}{n} \sum_{k=0}^{n} X(T^k w) \longrightarrow \mathbb{E}(X) \quad \text{a.s. and in mean}$$

$X = 1_A$



In effect: Time Average $\longrightarrow$ Space Average.

If we have a stationary sequence of random Variables $X_1, X_2, \ldots$, and if the shift transformation is Ergodic, then

$$\frac{1}{n} \sum_{k=0}^{n-1} \hat{X}_k \longrightarrow \mathbb{E}(X) \quad \text{a.s. and in mean}$$

$$\frac{1}{n} \sum_{k=0}^{n-1} X_k \longrightarrow \mathbb{E}(X) \quad \text{a.s and in mean}$$

Def$^{\underline{n}}$: An event $A \in (\Omega, \mathcal{F}, \mathbb{P})$, $A \in \mathcal{F}$ i.e., is invariant if $\exists$ $B \in \mathcal{B}(\mathbb{R}^\infty)$ s.t. $A = \{\omega : (X_n(\omega), X_{n+1}(\omega), \dots) \in B\}$ for all $n \geq 1$.,

where $X_1, X_2, \dots$ is a stationary seq$^{\underline{n}}$ of rvs.

- Check that the collection $\mathcal{I}$ of all invar. events defined as above is a σ-algebra.

Def$^{\underline{n}}$: A stationary process is ergodic if for every $A \in \mathcal{I}$, where $\mathcal{I}$ is the invariant σ-algebra assoc with the process, we have $\mathbb{P}(A) = 0$ or 1.

Remark.
We have shown that MPT $\longrightarrow$ Stationary Process and Stationary Process $\longrightarrow$ MPT in previous prop$^{\underline{n}}$s.

Thm: Let $X_1, X_2, \dots$ be a Stationary process and $\mathcal{I}$ the invariant σ-algebra. If $\mathbb{E}|X_1| < \infty$, then

$$\frac{1}{n}\sum_{i=1}^{n} X_i \longrightarrow \mathbb{E}(X_1 | \mathcal{I})$$ a.s. and in mean, and if the process is Ergodic,

$$\frac{1}{n}\sum_{i=1}^{n} X_i \longrightarrow \mathbb{E} X_1$$ as and in mean.

[Original proof by Kingman] $\rightarrow$ diff. variant.

Thm: (Subadditive Ergodic thm) [Liggett's Subadditive Ergodic Thm]

Suppose $\{X_{m,n}\}_{m \leq n}$ are random variables satisfying

a) $X_{0,0} = 0$ and $X_{0,n} \leq X_{0,m} + X_{m,n}$ $\forall\ 0 \leq m \leq n$

b) $\{X_{(n-1)k, nk} : n \geq 1\}$ is a stationary process $\forall\ k \geq 1$

c) $\{X_{m,m+k} : k \geq 0\}$ has the same distr$^{\underline{n}}$ as $\{X_{m+1, m+k+1}, k \geq 0\}$ $\forall\ m \geq 0$

d) $\mathbb{E} X_{0,1}^{+} < \infty$

Let $\alpha_n = \mathbb{E}(X_{0,n}) < \infty$ because

$$\mathbb{E} X_{0,n} \overset{(a)}{\leq} \mathbb{E} X_{0,1} + \mathbb{E} X_{1,n} \overset{(c)}{=} \mathbb{E} X_{0,1} + \mathbb{E} X_{0,n-1}$$, and

then use induction and (d). (Note $\mathbb{E} X_{0,1} = -\infty$ is possible)

Then, $$\alpha := \lim_{n \to \infty} \frac{\alpha_n}{n} = \inf_{n \geq 1} \frac{\alpha_n}{n} \in [-\infty, \infty)$$, and

$$X_\infty := \lim_{n \to \infty} \frac{X_{0,n}}{n}$$ exists a.s. and $X_\infty \in [-\infty, \infty)$ a.s.

Also, $\mathbb{E} X_\infty = \alpha$.

If $\alpha > -\infty$, then,

$$\lim_{n \to \infty} \mathbb{E}\left(\left| \frac{X_{0,n}}{n} - X_\infty \right| \right) = 0 \text{ , i.e., } \quad \frac{X_{0,n}}{n} \longrightarrow X_\infty \text{ in } L^1.$$

If the Stationary Processes in (b) are ergodic, then $X_\infty = \alpha$ a.s.

$$\begin{array}{ccccc}
X_{0,0} & X_{0,1} & X_{0,2} & X_{0,3} & \cdots \\
 & X_{1,1} & X_{1,2} & X_{1,3} & \cdots \\
 & & X_{2,2} & X_{2,3} & \cdots \\
 & & & X_{3,3} & \cdots \\
 & & & & \ddots
\end{array}$$

Lecture - 16

[Proof of the Subadditive Ergodic Thm]

03/10/24

Step 1: We have $X_{0,n} \leqslant X_{0,m} + X_{m,n}$

$$\Rightarrow \quad \mathbb{E} X_{0,n} \leqslant \mathbb{E} X_{0,m} + \mathbb{E} X_{m,n} \overset{(c)}{=} \mathbb{E} X_{0,m} + \mathbb{E} X_{0,n-m}$$

$$\therefore \quad \alpha_{m+n} \leqslant \alpha_m + \alpha_n \quad \forall\, m,n \qquad \left[\because X_{m,n} \overset{d}{=} X_{0,n-m}\right]$$

$\therefore$ By Fekete's Lemma, $\alpha = \lim_{n\to\infty} \frac{\alpha_n}{n} = \inf_{n\geqslant} \frac{\alpha_n}{n} \in [-\infty, \infty)$

Step 2: To show that $X_\infty := \lim_n \frac{X_{0,n}}{n}$ exists.

Let $\overline{X} := \limsup \frac{X_{0,n}}{n}$

Assume $\alpha > -\infty$ and fix $k \geqslant 1$. Then $X_{0,kn} \leqslant \sum_{j=1}^{n} X_{k(j-1),kj}$, using condition (a) multiple times. From condition (b), we know that $\{X_{k(j-1)}, X_{kj} : j \geqslant 0\}$ is a Stationary process. Let $\mathcal{I}_k$ be the invariant σ-algebra associated with the process. Then, by the Ergodic Thm,

$$\frac{1}{n}\sum_{j=1}^{n} X_{k(j-1),kj} \xrightarrow[L^1]{a.s.} \mathbb{E}\left(X_{0,k} \mid \mathcal{I}_k\right) \quad \text{as } n \to \infty.$$

Combining these, we get

$$\mathbb{E}\left(\limsup_n \frac{X_{0,kn}}{kn}\right) \leqslant \mathbb{E}\left(\limsup_n \frac{1}{k}\left(\frac{\sum_{j=1}^{n} X_{k(j-1),kj}}{n}\right)\right)$$

$$= \frac{1}{k}\,\mathbb{E}\left(\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} X_{k(j-1),kj}\right) = \frac{\mathbb{E}\left[\mathbb{E}\left[X_{0,k} \mid \mathcal{I}_k\right]\right]}{k}$$

$$= \frac{\mathbb{E}\left[X_{0,k}\right]}{k}.$$

So, along multiples of kn, we have gotten the upper bounds on mean. Again,

$$X_{0,kn+j} \leqslant X_{0,kn} + X_{kn,kn+j}$$

and so, we need to bound $X_{kn,kn+j}$. But by property (c), $X_{kn,kn+j} \overset{d}{=} X_{0,j}$. In particular, $\frac{X_{kn,kn+j}}{n} \overset{d}{=} \frac{X_{0,j}}{n}$. Thus,

$$\mathbb{E}\left(\limsup_{n\to\infty} \frac{X_{kn,kn+j}}{n}\right) \leqslant 0 \qquad (0 \text{ or } -\infty).$$

for any fixed j, and in particular, for $1 \leq j \leq k$.

$(N = kn + j)$

$$\therefore \quad \overline{X} = \limsup_N \frac{X_{0,N}}{N} \overset{(a)}{\leq} \limsup_n \left(\frac{X_{0,kn}}{kn+j} \right) + \max_{1 \leq j \leq k} \left\{ \limsup_n \frac{X_{kn,kn+j}}{kn+j} \right\}$$

$$\leq \limsup_{n \to \infty} \frac{X_{0,kn}}{kn} + \max_{1 \leq j \leq k} \limsup_{n \to \infty} \frac{X_{kn,kn+j}}{kn}$$

$$\leq \limsup_{n \to \infty} \frac{X_{0,kn}}{kn}$$

$$\therefore \quad \mathbb{E}\,\overline{X} \leq \frac{\alpha_k}{k} \quad (\forall\, k \in \mathbb{N}\,!)$$

In particular, $\mathbb{E}\,\overline{X} \leq \inf_{k \geq 1} \frac{\alpha_k}{k} = \alpha$

$\rightarrow$ If the Stationary process $\{X_{k(j-1),kj}\}$ is Ergodic, then

$$\frac{1}{n} \sum_{j=1}^{n} X_{k(j-1),kj} \xrightarrow[L^1]{a.s.} \alpha_k$$

$$\Rightarrow \quad \overline{X} \leq \alpha \quad \text{a.s. just as before.}$$

<u>Step 3</u>: Let $\underline{X} = \liminf_n \frac{X_{0,n}}{n}$.

Assume $\alpha > -\infty$.

Let $U_n \sim \text{Unif}\{1, \ldots, n\}$ and $\{U_n : n \geq 1\}$ a collection of indep r.v.s.

Define $Y_n^k = X_{0,k+U_n} - X_{0,k+U_n-1}$

$$\therefore \mathbb{E}\,Y_n^k = \mathbb{E}\,(X_{0,k+U_n} - X_{0,k+U_n-1})$$

$$= \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}\,(X_{0,k+j}) - \mathbb{E}\,(X_{0,j+k-1})$$

$$= \frac{1}{n}\, \mathbb{E}[X_{0,k+n}] - \mathbb{E}[X_{0,k}] \qquad \text{(Telescoping sum)}$$

$$\geq \frac{n+k}{n} \left(\alpha - \frac{\alpha_k}{n} \right) \qquad \left[\because\ \alpha = \inf \frac{\alpha_l}{l}\right]$$

$$> -\infty \qquad \text{[By Assumption]}$$

Again,
$$\mathbb{E}\,(Y_n^k)^+ = \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}\,(X_{0,k+j} - X_{0,k+j-1})^+ \quad (a)$$

$$\leq \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}\,(X_{k+j-1,\,k+j})^+ \quad (c)$$

$$= \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}\,X_{0,1}^+ = \mathbb{E}\,X_{0,1}^+ < \infty$$

(by cond (d))

$\therefore \sup_n \mathbb{E}|Y_n^k| < \infty$ from the previous two bounds, $\forall k \in \mathbb{N}$.

$\longrightarrow$ ⊛

Thus, ⊛ says that $\{Y_n^k : k \geq 0\}$ is tight.

Digression: <u>Tightness</u>:

Tightness of $\{\{Z_n^k : n \geq 1\} : k \geq 1\}$ means that $\exists$ a subsequence $\{n_i : i \geq 1\}$ with $n_i \longrightarrow \infty$ as $i \to \infty$ s.t. the joint distr$\underline{n}$ of $\{Z_{n_i}^k : k \geq 0\}$ converges to that of some collection $\{Y_k : k \geq 0\}$

<u>Claim</u>: The tight sequence $\{Y_k : k \geq 0\}$ is a stationary process, i.e., we need to show $(Y_0, Y_1, \ldots, Y_m) \overset{d}{=} (Y_1, \ldots, Y_{m+1})$ for any $m \geq 0$.

- Conv in distr$\underline{n}$ of a sequence of random vectors:

  - $X_n \xrightarrow{d} X \iff F_n(x) \longrightarrow F(x) \;\; \forall x \in \text{Cont}(F)$

    $\iff \mathbb{P}(X = x) = 0$.

  - For a r. vec., $(Y_0, \ldots, Y_m)$,

    $\updownarrow$

    we have a cont set, i.e., a set C s.t. $\mathbb{P}((Y_0, \ldots, Y_m) \in \partial C) = 0$ and the convergence is distribution of $\{(Y_0^n, Y_1^n, \ldots, Y_m^n) : n \geq 1\}$ to $(Y_0, \ldots, Y_m)$ means.

$$\mathbb{P}((Y_0^n, \ldots, Y_m^n) \in C) \xrightarrow{n \to \infty} \mathbb{P}((Y_0, \ldots, Y_m) \in C) \quad \forall C \text{ cont sets.}$$

<u>Proof of Claim</u>: Let C be a cont. set of $(Y_0, \ldots, Y_m)$

Now,
$$\mathbb{P}((Y_0, \ldots, Y_m) \in C) = \lim_{i \to \infty} \mathbb{P}((Y_0^{n_i}, \ldots, Y_m^{n_i}) \in C)$$

$$= \lim_{i \to \infty} \mathbb{P}((X_{0,U_{n_i}} - X_{0,U_{n_i}-1}, \ldots, X_{m,U_{n_i}} - X_{m,U_{n_i}-1}) \in C)$$

$$= \lim_{i \to \infty} \sum_{j=1}^{n_i} \mathbb{P}((X_{0,j} - X_{0,j-1}, \ldots, X_{m,j} - X_{m,j-1}) \in C \mid U_{n_i} = j)\, \mathbb{P}(U_{n_i} = j)$$

$$= \lim_{i \to \infty} \left[\sum_{j=2}^{n_i} \frac{1}{n_i} \mathbb{P}((\;) \in C \mid U_{n_i} = j) + \frac{1}{n_i} \mathbb{P}((\;) \in C \mid U_{n_i} = 1) \right]$$

Take $j=1$ here

$$= \lim_{i \to \infty} \frac{1}{n_i} \sum_{j=2}^{n_i} \mathbb{P}((\;) \in C \mid U_{n_i} = j)$$

$$= \lim_{i\to\infty} \frac{1}{n_i} \sum_{t=1}^{n_i - 1} \mathbb{P}\left((X_{0,1+t} - X_{0,t}, \dots, X_{m,1+t} - X_{m,t}) \in C \,\middle|\, U_{n_i} = t\right)$$

$$= \lim_{i\to\infty} \frac{1}{n_i} \sum_{t=1}^{n_i} \mathbb{P}\left((X_{0,1+t} - X_{0,t}, \dots, X_{m,1+t} - X_{m,t}) \in C \,\middle|\, U_{n_i} = t\right)$$

[Adding a term d.n. change the limit]

$$= \lim_{i\to\infty} \mathbb{P}\left((X_{0,1+U_{n_i}} - X_{0,U_{n_i}}, \dots, X_{m,1+U_{n_i}} - X_{m,U_{n_i}}) \in C\right)$$ [Unconditioning]

$$= \lim_{i\to\infty} \mathbb{P}\left((Y_1^{n_i}, \dots, Y_{m+1}^{n_i}) \in C\right) = \mathbb{P}((Y_1, \dots Y_{m+1}) \in C).$$ ▨

<u>Defn:</u> $\{X_n : n \geq 1\}$ is uniformly integrable if $\displaystyle\lim_{\alpha\to\infty} \sup_n \int_{\{|X_n| \geq \alpha\}} |X| \, d\mathbb{P} = 0$.

Now, $(Y_1^n)^+ = (X_{0,1+U_n} - X_{0,U_n})^+ \leq (X_{U_n, 1+U_n})^+ \overset{d}{=} X_{0,1}^+$

Then $\{(Y_1^n)^+ : n \geq 1\}$ is U.I. because

$$0 \leq \lim_{\alpha\to\infty} \sup_n \int_{\{(Y_1^n)^+ \geq \alpha\}} (Y_1^n)^+ d\mathbb{P} \leq \lim_{\alpha\to\infty} \int_{\{X_{0,1}^+ \geq \alpha\}} X_{0,1}^+ \, d\mathbb{P} = 0$$

The last ineq. follows by DCT as $X_{0,1}^+ \cdot 1_{\{X_{0,1}^+ \geq \alpha\}} \downarrow 0$.

<u>H.W.</u>

⊛ <u>Fatou's Lemma:</u>

a) If $\{X_n : n \geq 1\}$ is U.I. from above, i.e., $\{X_n^+ : n \geq 1\}$ is U.I., then

$$\mathbb{E}(\limsup_n X_n) \geq \limsup_n \mathbb{E}(X_n).$$

b) If $\{X_n : n \geq 1\}$ is U.I. from below, then

$$\liminf \mathbb{E} X_n \geq \mathbb{E}(\liminf X_n).$$

Now, $\mathbb{E}(Y_1) \geq \limsup_{i} \mathbb{E}(Y_1^{n_i}) \geq \liminf_{i\to\infty} \mathbb{E}(Y_1^{n_i})$ [By Fatou's Lemma]

$\geq \alpha$

Now, $\left(X_{0,k+1} - X_{0,k}, \ldots, X_{0,k+j} - X_{0,k} \right)$

$$\le \left(X_{k,k+1}, \ldots, X_{k,k+j} \right) \overset{d}{=} \left(X_{0,1}, \ldots, X_{0,j} \right)$$

coordinatewise pointwise ineq.

As (Y_k) is stationary

Note: $\lim_n \frac{1}{n} \sum_{i=0}^{n-1} Y_i$ exists almost surely by the Ergodic thm, and $\mathbb{E} \frac{1}{n} \sum_{i=0}^{n-1} Y_i = \mathbb{E} Y_1$

For $f : \mathbb{R}^j \to \mathbb{R}$, bounded cont, non-decreasing,

$$\mathbb{E}\, f(X_{0,k+1} - X_{0,k}, \ldots, X_{0,k+j} - X_{0,k}) \le \mathbb{E}\, f(X_{0,1}, \ldots, X_{0,j})$$

$$\mathbb{E}\, f(Y_0, Y_0 + Y_1, \ldots, Y_0 + Y_1 + \ldots + Y_{j-1})$$

Take $f(x_1, \ldots, x_j) = \frac{1}{j} x_j$. Then,

$$\lim \frac{1}{n} \sum_{i=0}^{n-1} Y_i \overset{d}{\preccurlyeq} \underline{X} \qquad \left[\because \underline{X} = \liminf_{j \ge 1} \frac{X_{0,j}}{j} \right]$$

$\therefore \mathbb{E}\, \underline{X} \ge \alpha$

Combining, we get $X_\infty = \lim \frac{X_{0,n}}{n}$ exists, and $\mathbb{E} X_\infty = \alpha$.

$$\left[\because \mathbb{E}\, \underline{X} \ge \alpha \ge \mathbb{E}\, \bar{X} \Rightarrow \mathbb{E}(\underline{X} - \bar{X}) \ge 0. \text{ But } \underline{X} - \bar{X} \le 0 \Rightarrow \underline{X} = \bar{X} \text{ a.s.} \right]$$

Again, if $\{X_{k(j-1), kj}\}_{j \ge 0}$ is Ergodic then $\{Y_k\}_{k \ge 0}$ is also Ergodic, and thus $\underline{X} \ge \alpha$. Combining with the prev. step, we get X_∞ exists and $X_\infty = \alpha$ a.s..

Step 4: $\frac{X_{0,n}^+}{n} \le \frac{1}{n} (X_{0,1} + \ldots + X_{n-1,n})^+ \le \frac{1}{n} (X_{0,1}^+ + \ldots + X_{n-1,n}^+)$

Check: $\left\{ \frac{X_{0,n}^+}{n} : n \ge 1 \right\}$ is U.I..

$$|z| = z^+ + z^- = 2z^+ - (z^+ - z^-) = 2z^+ - z$$

[H.W.] $\Rightarrow \lim_n \mathbb{E} \left(\frac{X_{0,n}}{n} - X_\infty \right)^+ = 0$

$$\Rightarrow \mathbb{E}\left(\left| \frac{X_{0,n}}{n} - X_\infty \right| \right) = 2\, \mathbb{E} \left(\frac{X_{0,n}}{n} - X_\infty \right)^+ - \mathbb{E}\left(\frac{X_{0,n}}{n} - X_\infty \right) \longrightarrow 0$$

Step 5

Final Step: $\alpha = -\infty$. Take

$$X^N_{m,n} = \max\{X_{m,n}, -N(n-m)\}$$

Check: For fixed N, $\{X^N_{m,n} : 0 \le m \le n\}$ satisfies the conditions of the Theorem and in addition $\alpha^N := \inf_{m \ge 1} \frac{1}{m} \mathbb{E}(X^N_{0,m}) \ge -N > -\infty$

for each $N \in \mathbb{N}$.

H.W. Complete Step 1 – Step 4 for $\{X^N_{m,n} : 0 \le m \le n\}$

$$\frac{1}{n} X^N_{0,n} = \max\{\frac{1}{n} X_{0,n}, -N\}$$

And $\frac{X^N_{0,n}}{n} \longrightarrow X^N_\infty$.

$\boxed{\text{Homework}}$ Fatou's Lemma for U.I. from above / below.

b) Hint: i) Given $\varepsilon > 0$ by U.I., get $C > 0$ s.t.

$$\mathbb{E}\left(X_n^- 1_{\{X_n^- > C\}}\right) < \varepsilon \quad \forall n.$$

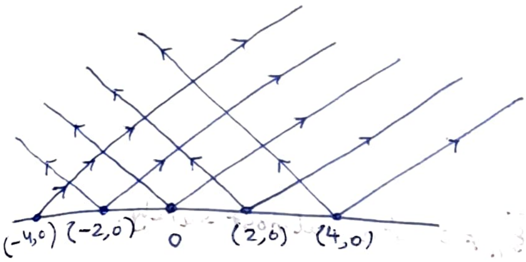
(ii) $X + C \le \liminf (X_n + C)^+$

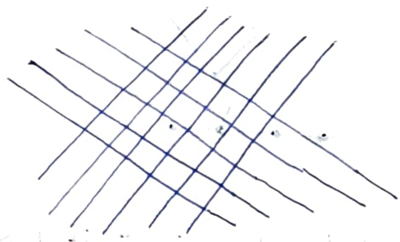
(iii) Use the fact that $(X_n+C)^+ = (X_n+C) + (X_n+C)^-$
$\le (X_n+C) + X_n^- 1_{\{X_n^- \ge C\}}$.

Lecture-17

Oriented Percolation

- $\mathcal{V} = \mathbb{Z}^2_{even} = \{(m,n) \mid m,n \in \mathbb{Z}, m+n \in 2\mathbb{Z}\}$
- $\mathcal{E} = \{ \overrightarrow{(x,y)} : x = (m,n) \in \mathbb{Z}^2_{even}, \ y = (m+1, n+1) \text{ or } y = (m-1, n+1), n \geq 0 \}$





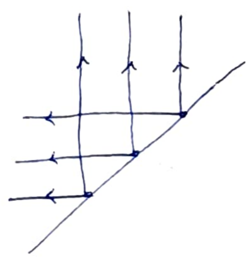
- A path $\pi = (x_0, x_1, \ldots)$ s.t. $x_0, x_1, \ldots \in \mathbb{Z}^2_{even}$ and $(x_i, x_{i+1}) \in \mathcal{E} \ \forall i$
- An edge is declared open (1) or closed (0) independent of other edges with probability p or $1-p$ respectively. So, we are working on the prob. space $(\{0,1\}^{\mathcal{E}}, \mathcal{F}, \mathbb{P}_p)$, where $\mathcal{F}$ is the σ-algebra generated by cylinder sets and $\mathbb{P}_p$ is the product measure of Bernoulli measures.

- A path π is open if all its edges are open.
- $\underset{\sim}{u}, \underset{\sim}{v} \in \mathbb{Z}^2_{even}$

$$\{\underset{\sim}{u} \longrightarrow \underset{\sim}{v}\} = \{\exists \text{ an open path from } \underset{\sim}{u} \text{ to } \underset{\sim}{v}\}$$

$$C(\underset{\sim}{0}) = \{\underset{\sim}{u} \in \mathbb{Z}^2_{even} : 0 \longrightarrow u\}$$

$$p_c = \inf \{p : \mathbb{P}_p(\# C(\underset{\sim}{0}) = \infty) > 0\}$$

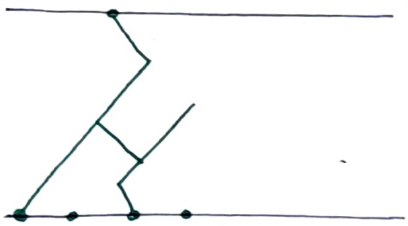


Remark

- Rotating the $(\mathbb{Z}^2_{even}, \mathcal{E})$ lattice by 45°, we have. Here the edges are directed.

- In particular, $p_c(\text{oriented}) \geq p_c(\mathbb{Z}^2)$.

Let $\bar{\xi}_n = \{ m \in \mathbb{Z} \mid (m,n) \in \mathbb{Z}^2_{even}$ and $\exists (k,0) \in \mathbb{Z}^2_{even}$, $k \le 0$ s.t. $(k,0) \xrightarrow{open} (m,n) \}$



$\bar{r}_n := \sup\{\bar{\xi}_n\}$

<u>H.W.</u> Show that if $p > 0$, then $\bar{\xi}_n \neq 0 \ \forall n$. almost surely.

This H.W. tells us that we never need to worry about $\bar{r}_n = -\infty$.

<u>Prop$^{\underline{n}}$:</u> 1) $\xi_n^0 = \bar{\xi}_n \cap [l_n, \infty)$

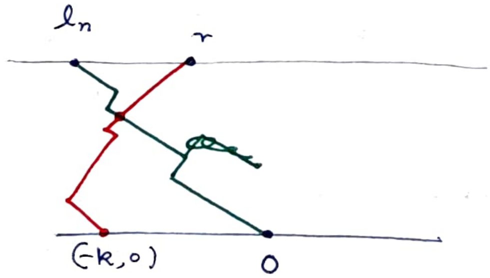
2) On the set $\{\xi_n^0 \neq \emptyset\}$, $r_n = \bar{r}_n$

~~Here~~

<u>Proof:</u> 1) Evidently, $\xi_n^0 \subseteq \bar{\xi}_n$ and $\xi_n^0 \subseteq [l_n, \infty) \Rightarrow \xi_n^0 \subseteq \xi_n \cap [l_n, \infty)$

Conversely, if $l_n = \infty$, we have $\xi_n^0 = \phi$. So there is nothing to prove. Suppose $l_n < \infty$.

Note that • $\exists$ a path (green) $0 \longrightarrow l_n$. If we can reach from a point strictly left of 0 to a point strictly to the right of l_n by a open path (red), then $0 \xrightarrow{open} r$ as the paths intersect. This gives a proof of (1)



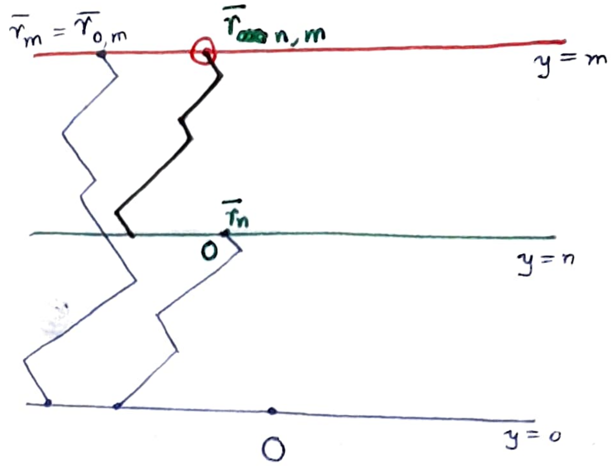
2) This follows directly from (1)

Thus on the set $\{\xi_n^0 \neq \emptyset\}$, it would suffice to study $\bar{r}_n$ instead of r_n

Take $n \le m$

Consider s.t. s.t. $\bar{r}_n < \infty$

We can then use this point as the origin for the rest of the process. With respect to this new origin, let



$$\bar{r}_{n,m} = \sup \left\{ x - \bar{r}_n \;\middle|\; \begin{array}{l} (x,m) \in \mathbb{Z}^2_{even} \ \& \\ \exists\, y_1 \le \bar{r}_n \text{ s.t.} \\ (y_1, n) \xrightarrow{\text{open}} (x,m) \end{array} \right\}$$

Note: $\bar{r}_n > -\infty$ as $\bar{\xi}_n \neq \phi$ wp 1.

- In this notation, $\bar{r}_{0,n} = \bar{r}_n$.

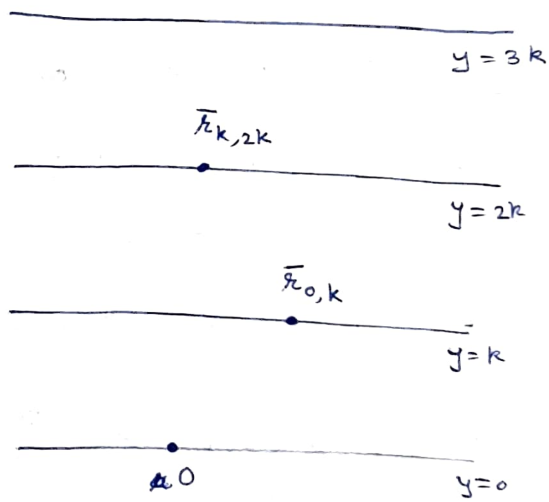
// We have:

1) $\bar{r}_{0,0} = 0$

2) $\bar{r}_{0,m} \le \bar{r}_{0,n} + \bar{r}_{n,m}$ for $0 \le n \le m$. [Use prev. propⁿ and defⁿ of $\bar{r}_{n,m}$]

Observe that

$\bar{r}_{0,k} \overset{d}{=} \bar{r}_{k,2k}$ and these are independent



This follows from the defⁿ of $\bar{r}_{n,m}$, which depends only on edges between $y = n$ and $y = m$.

$\therefore$ (3) $\{ \bar{r}_{(n-1)k,\, nk} \}_{n \in \mathbb{N}}$ is an iid seq of random variables.

Again, note the $\bar{r}_{0,k} \overset{d}{=} \bar{r}_{1,k+1} \overset{d}{=} \bar{r}_{2,k+2}$ (not necessarily independent). Thus,

(4) For every $m \ge 0$, $\{ r_{m,m+k} ; k \ge 0 \} \overset{d}{=} \{ r_{m+1,\, m+k+1} : k \ge 0 \}$

(5) $\bar{r}_{0,1} \le 1$ and $\inf \bar{r}_{0,1} = -\infty$

As iid sequences are stationary, $\{\bar{r}_{m,n}\}$ satisfies the properties of Liggett's Subadditive Ergodic Thm.

$\therefore$ $\dfrac{\bar{r}_{0,n}}{n} \longrightarrow \alpha$ (say) a.s., where $\alpha \in [-\infty, 1]$

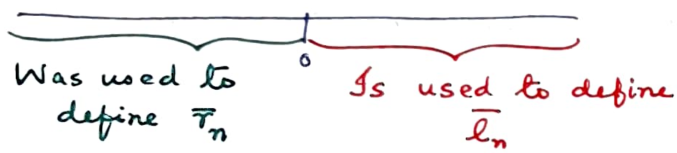
and $\alpha = \inf_n \dfrac{\mathbb{E}[\bar{r}_{0,n}]}{n}$

Thus, on the set $\{\# C(0) = \infty\}$, by our proposition,

$$\frac{r_n}{n} \longrightarrow \alpha \quad a.s.$$

$$\left[\because \ \{\# C(0) = \infty\} = \bigcap_{n \geq 1} \{\xi_n^0 \neq \emptyset\} \right]$$

- Similarly, we define $\bar{l}_n$ as the inf of $\left\{ m \,\middle|\, \begin{array}{l} (m,n) \in \mathbb{Z}^2_{even}, \text{ and } \exists\, (k,0) \in \mathbb{Z}^2_{even} \\ \text{with } k \geq 0 \text{ s.t.} \\ (k,0) \xrightarrow{open} (m,n) \end{array} \right\}$



H.W. Define $\bar{l}_{n,m}$ for $0 \leq n \leq m$ for this setup and show that the Subadditive Ergodic Thm holds for $\{\bar{l}_{n,m} \mid 0 \leq n \leq m, m \geq 0\}$

Note: This gives us $l_{0,m} \geq l_{0,n} + l_{n,m} \longrightarrow$ then we work with $-l_{m,n}$.

Also, by symmetry, $\frac{\bar{l}_{0,n}}{n} \xrightarrow{a.s.} -\alpha$, where α is exactly as before!
on the set $\{\# C(0) = \infty\}$.

If $\mathbb{P}_p(\# C(0) = \infty) > 0$, then we must have $\alpha \geq -\alpha$, i.e., $\alpha \geq 0$.

(★) Thm: If $\alpha^{(p)} < 0$, then $\mathbb{P}_p(\# C(0) = \infty) = 0$.

We now go to the proof of the following, which complements this thm.

(★) Thm: If $\alpha(p) > 0$, then $\mathbb{P}_p(\# C(0) = \infty) > 0$.

Some Definitions:

- For $A \subseteq (-\infty, \infty)$, let

$$\xi_n^A = \{m : (m,n) \in \mathbb{Z}^2_{even}, \ \exists\, k \in A, \ (k,0) \xrightarrow{open} (m,n)\}$$

$$r_n^A = \sup \xi_n^A, \quad l_n^A = \inf \xi_n^A \quad \text{with the usual conventions.}$$

$$\tau^A = \inf\{t : \xi_t^A = \emptyset\}$$

H.W. Take $M > 0$.

Show that
$$\xi_n^{[-M,M]} = \xi_n^{(-\infty,M]} \cap \left[l_n^{[-M,M]}, \infty\right]$$
$$= \xi_n^{[-M,\infty)} \cap \left[-\infty, r_n^{[-M,M]}\right]$$
$$= \xi_n^{(-\infty,\infty)} \cap \left[l_n^{[-M,M]}, r_n^{[-M,M]}\right]$$

Lecture-18

Today, we prove the following theorem:

Thm: If $\alpha > 0$, then $\mathbb{P}_p(\# C(0) = +\infty) > 0$.

Proof: Recall from the H.W. given in the previous class,

$$\xi_n^{[-M,M]} = \xi_n^{(-\infty,M]} \cap [\ell_n^{[-M,M]}, \infty)$$

$$= \xi_n^{[-M,\infty)} \cap (-\infty, r_n^{[-M,M]}]$$

$$= \xi_n^{(-\infty,\infty)} \cap [\ell_n^{[-M,M]}, r_n^{[-M,M]}]$$

In particular, ~~if~~ on $\{\xi_n^{(-\infty,M]} \neq \phi\}$, we have

$$r_n^{[-M,M]} = r_n^{(-\infty,M]}, \quad \ell_n^{[-M,M]} = \ell_n^{[-M,\infty)}$$

$$\therefore \quad \tau^{[-M,M]} = \inf\{n : r_n^{[-M,M]} < \ell_n^{[-M,M]}\}$$

$$= \inf\{n : r_n^{(-\infty,M]} < \ell_n^{[-M,\infty)}\}$$

In other words, $\{\ell_n^{[-M,\infty)} \leq 0 \leq r_n^{(-\infty,M]} \ \forall n\} \subseteq \{\tau^{[-M,M]} = \infty\}$

By translation invariance,

$$\mathbb{P}_p(r_n^{(-\infty,M]} > 0 \ \forall n) = \mathbb{P}_p(\bar{r}_n = r_n^{(-\infty,0]} > -M \ \forall n)$$

Now we state a useful claim.

Claim: If $\alpha > 0$, i.e., $\frac{\bar{r}_n}{n} \longrightarrow \overset{>0}{\alpha}$, then for any $\eta > 0$, $\exists\ M = M(\eta)$ s.t.

$$\mathbb{P}_p \left(\bar{r}_n > -M \quad \forall n \in \mathbb{N} \right) \geq 1 - \eta$$

Assuming this claim, we have:

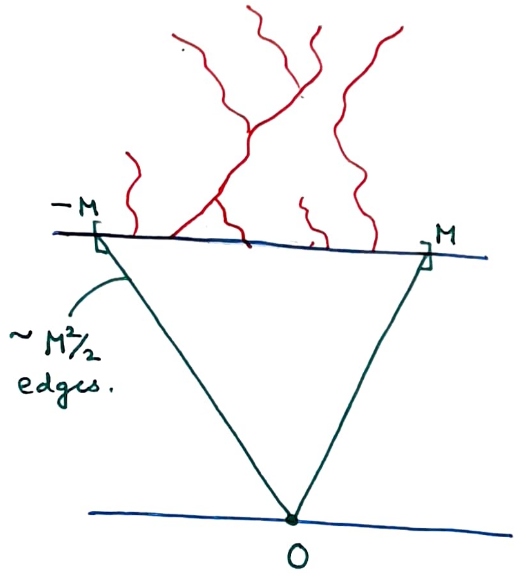
(Inclusion Exclusion Principle)

$$\mathbb{P}_p \left(\tau^{[-M,M]} = \infty \right) \geq \mathbb{P}_p \left(r_n^{(-\infty, M)} > 0 \ \forall n \right) + \mathbb{P}_p \left(\ell_n^{[-M,\infty)} < 0 \ \forall n \right) - 1$$

$$\geq (1-\eta) + (1-\eta) - 1 = 1 - 2\eta. > 0 \text{ if } \eta < \tfrac{1}{2}.$$

Thus the process started from $[-M,M]$ survives with positive probability. Again,

$$\mathbb{P}_p \left(\# C(0) = \infty \right) \geq p^{M^2/2} (1-2\eta) > 0$$

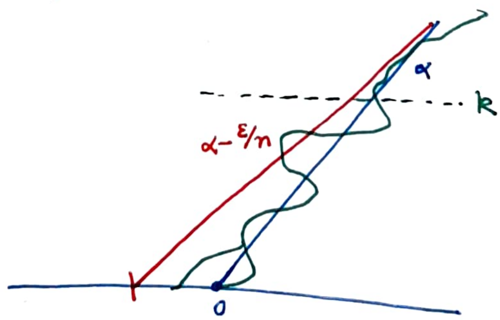


Thus assuming the claim, we have $\alpha > 0 \Rightarrow$ process percolates.

Proof of Claim: Fix $\varepsilon > 0$.

$$B_k := \{ \omega : \bar{r}_n(\omega) \geq \alpha n - \varepsilon \quad \forall n \geq k \}$$

If $\frac{\bar{r}_n(\omega)}{n} \geq \alpha - \frac{\varepsilon}{n}$, the path to $\bar{r}_n$ cannot cross the red barricade.



Thus,

$$B_k = \left\{ \omega : \text{path } \overset{[-\infty,0)}{\bullet} \longrightarrow \bar{r}_n(\omega) \text{ does not cross the barricade after } n \geq k \right\}$$

Now, $\frac{\bar{r}_n}{n} \longrightarrow \alpha$ as $n \longrightarrow \infty$, $|\bar{r}_n n - n\alpha| < \varepsilon$

$\therefore$ If $B_k \uparrow \Omega'$, $\mathbb{P}(\Omega') = 1$.

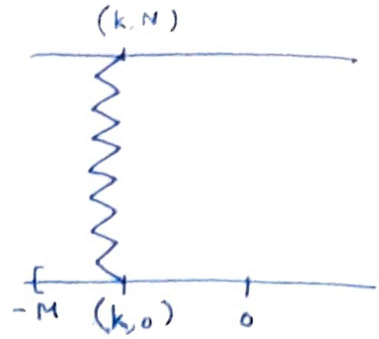
So given $\eta > 0$, $\exists\ N$ s.t. $\mathbb{P}_p(B_n) > 1 - \frac{\eta}{2} \quad \forall n \geq N$.

Assume WLOG that N is even and $\alpha n - \varepsilon \geq 1 \quad \forall n \geq N$.

$$\therefore \mathbb{P}_p \left(\bar{r}_n \geq \alpha n - \varepsilon \quad \forall n \geq N \right) > 1 - \frac{\eta}{2}$$

For k even, let $E_{k,N} = \{\overrightarrow{(k,0),(k-1,1)};\ \overrightarrow{(k-1,1),(k,2)}, \dots, \overrightarrow{(k-1,N-1),(k,N)}\}$

be the zig-zag path from $(k,0)$ to (k,N).



$\therefore\ \mathbb{P}_p(\text{All edges of } \overset{\mathcal{E}_{k,N}}{E_{k,N}} \text{ are open}) = p^N$

$\therefore\ \mathbb{P}(\text{None of } \mathcal{E}_{0,N}, \mathcal{E}_{-2,N}, \dots, \mathcal{E}_{-M,N} \text{ occur})$

$= (1-p^N)^{M/2} < \frac{\eta}{2}$ for all M suff large.

Let $\Omega_1 = \{\overline{r}_n \geq \alpha n - \varepsilon \quad \forall\, n \geq N\}$

$\Omega_2 = \{\text{None of } \mathcal{E}_{0,N}, \mathcal{E}_{-2,N}, \dots, \mathcal{E}_{-M,N} \text{ occur}\}$

Then on $\Omega_1 \cap \Omega_2^c$, we have

$\overline{r}_n \geq -M$ and $\overline{r}_n \geq \alpha n - \varepsilon \quad \forall\, n \geq N,$

and $\mathbb{P}(\Omega_1 \cap \Omega_2^c) \geq 1 - \eta$

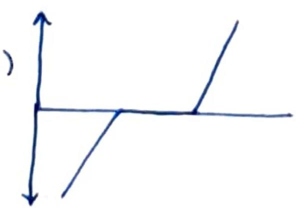
This completes the proof of the claim. ▨

- Coupling Arguments show that $\alpha(p)$ is non-decreasing in p.

– We have shown

$$\sup\{p : \alpha(p) < 0\} \leq p_c \leq \inf\{p : \alpha(p) > 0\}$$

However, it could be the case that $\{p : \alpha(p) = 0\}$ is a non-trivial interval. We now need to rule this out.



✪ **Propⁿ:** $p_c = \inf\{p : \alpha(p) > 0\}$

$p_c = \sup\{p : \alpha(p) < 0\}$

Proof:

We will show that if $\alpha(p') > -\infty$, then

$\alpha(p) - \alpha(p') \geq 2(p - p')$ for $p > p'$.

Step 1: For $C, D \in 2\mathbb{Z}$,

$\xi_n^{C \cup D} = \xi_n^C \cup \xi_n^D$

So, i) $r_n^{C\cup D} = \max\{r_n^C, r_n^D\}$

ii) $r_n^{C\cup D} - r_n^C = \max\{0, r_n^D - r_n^C\} = (r_n^D - r_n^C)^+$

So there is some sort of -ve correlation

Take $A \supseteq B$, both infinite subsets of -ve even integers.

$$\therefore\ r_n^{B\cup\{0\}} - r_n^B = \left(r_n^{\{0\}} - r_n^B\right)^+ \geq \left(r_n^{\{0\}} - r_n^A\right)^+ = r_n^{A\cup\{0\}} - r_n^A$$

As $A \supseteq B$, $r_n^A \geq r_n^B$

$$\therefore\ \mathbb{E}_p\left(r_n^{\{0,-2,-4,\dots\}} - r_n^{\{-2,-4,\dots\}}\right) = 2 \quad \text{[By transl invariance]}$$

$$\therefore\ \mathbb{E}_p\left(r_n^{B\cup\{0\}} - r_n^B\right) \geq \mathbb{E}_p\left(r_n^{A\cup\{0\}} - r_n^A\right) = 2$$

<u>Note</u>: We needed A, B to be infinite subsets as o.w. $r_n^A = -\infty$ with positive probability.

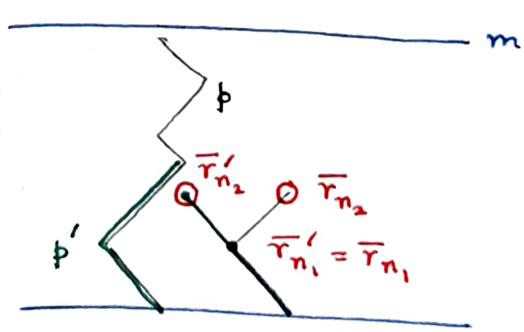
<u>Step 2</u>: $p > p'$ and $\alpha_n := \mathbb{E}_p(\bar{r}_n)$

$$\Rightarrow \alpha_n(p) - \alpha_n(p') \geq 2\left[1 - (1-(p-p'))^n\right]$$

We construct both the p and p' process on the same prob. space by the usual Coupling argument. Let $\{U_e\} \overset{iid}{\sim} \mathrm{Unif}[0,1]$, with configurations ω and ω' generated as usual

$$\omega(e) = 1_{U_e \leq p}$$
$$\omega'(e) = 1_{U_e \leq p'}$$

Let $\bar{r}_n$ and $\bar{r}_n'$ be the location of the rightmost edges at time n of the p and p' processes respectively.

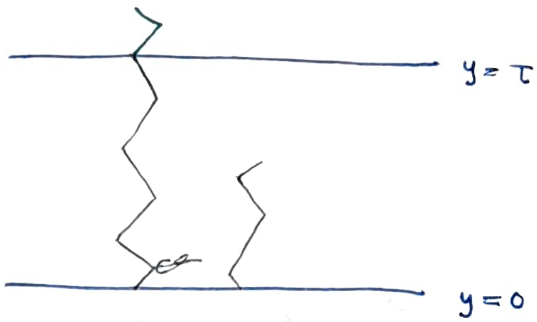


Let $\tau = \inf\{n : \bar{r}_n > \bar{r}_n'\}$ be the first time the random variables corr. to right-most points are different.

Random ofc.

We have $\bar{\xi}_n$ and $\bar{\xi}_n'$ as usual. We now construct $\bar{\xi}_n''$.

- $\bar{\xi}_n''$ is constructed with paramet. p in the region $0 \le y \le \tau$, whereas in the region $y \geq \tau$, it is constructed with paramet. p'.



Observe:

(i) $\bar{r}_\tau = \bar{r}_\tau'' \geq \bar{r}_\tau' + 2$

(ii) $\bar{\xi}_\tau = \bar{\xi}_\tau'' \supseteq \bar{\xi}_\tau' \cup \{\bar{r}_\tau\}$ (there may be more points ofc)

Now, note that $\left(\bar{\xi}_{n+1} \,|\, \bar{\xi}_n\right) \stackrel{d}{=} \left(\bar{\xi}_{n+1} \,|\, \bar{\xi}_n, \bar{\xi}_{n-1}, \dots, \bar{\xi}_0\right)$.

In particular, $\{\bar{\xi}_n\}$ is a Markov Process taking values in subsets of $\mathbb{Z}$. Thus, by the strong Markov Property,

$$\mathbb{E}_p(\bar{r}_n) - \mathbb{E}_p(\bar{r}_n') \geq \mathbb{E}_p(\bar{r}_n'') - \mathbb{E}_p(\bar{r}_n')$$
$$\geq 2\, \mathbb{P}_p(\tau \le n)$$
$$\geq 2\left[1 - (1 - (p - p'))^n\right]$$

The last step follows as at each ~~step~~ age, we have a bifurcation with probability $p - p'$.

Bifurcation $\equiv$
- $\bar{r}_{n+1} = \bar{r}_n + 1$
- $\bar{r}_{n+1}' \le \bar{r}_n' + 1$

This completes the proof of step 2. $\square$

<u>Step 3:</u> Fix $M \geq 1$ and take $\delta = \frac{p - p'}{M}$. Then

$$\alpha_n(p) - \alpha_n(p') = \sum_{m=1}^{Mn} \left(\alpha_n\left(p' + \frac{m\delta}{n}\right) - \alpha_n\left(p' + \frac{(m-1)\delta}{n}\right)\right)$$
$$\geq 2 \sum_{m=1}^{Mn} \left(1 - \left(1 - \frac{\delta}{n}\right)^n\right) \geq 2Mn\left(1 - \left(1 - \frac{\delta}{n}\right)^n\right)$$

Now we divide both sides by n and let $n \to \infty$.

$$\therefore \quad \alpha(p) - \alpha(p') \geq 2M\left(1 - e^{-\frac{(p-p')}{M}}\right)$$
$$\longrightarrow 2(p - p') \text{ as } M \to \infty. \quad \square$$

Note: Here, as $\alpha(p) \geq \alpha(p') > -\infty$, the conv. in mean of the Subadditive Ergodic Thm holds.