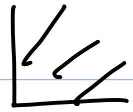
Till now : Percolation on $\mathbb{Z}^d$

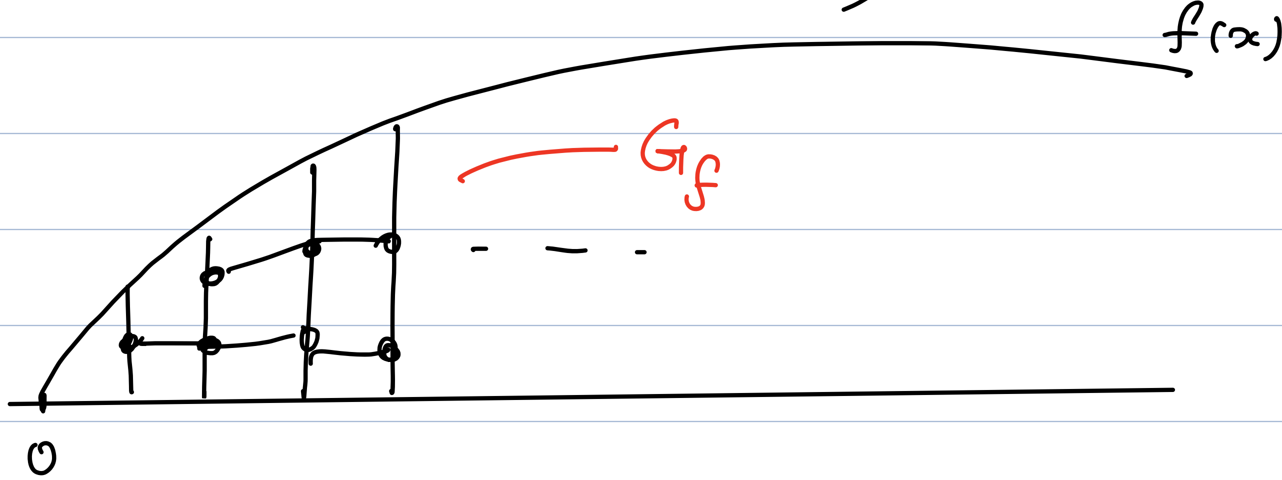
What about $(\mathbb{Z}^2)_+$ or $\mathbb{H}$



We will now focus on percolation on subsets of $\mathbb{Z}^2$.

Let $f : [0,\infty) \longrightarrow [0,\infty)$

$$G_f = \left\{ (x,y) \in \mathbb{Z}^2 : \quad 0 \le y \le f(x) \right\}$$



$$p_c(G_f) = \sup \left\{ p : \mathbb{P}_p \left(G_f \text{ admits an unbdd open component} \right) = 0 \right\}$$

Exercise : By 0-1 law if $p > p_c$ then w.p. 1 $\exists$ an unbounded open component.

Thm : Let $a > 0$, and $f : [0,\infty) \longrightarrow [0,\infty)$ if $\dfrac{f(x)}{\;} \longrightarrow a$ then $p_c(G_f)$ is the unique

$\log x$

Solution of the equation $\xi(1-p) = a$ where ξ is the correlation length.

Remark

① $\xi(1-p) = a$ has an unique solution

② If $f(x) = o(\log x)$ then $p_c(G_f) = 1$ and if $\log x = o(f(x))$ then $p_c(G_f) = \frac{1}{2}$.

Proof: First let $p > \frac{1}{2}$, otherwise $\xi(1-p) = \infty$ (meaningless)

Suppose $a < \xi(1-p)$, get $\delta > 0$ s.t. $a(1+\delta) < \xi(1-p)$

We'll show that G_f does not admit any unbounded open component a.s.

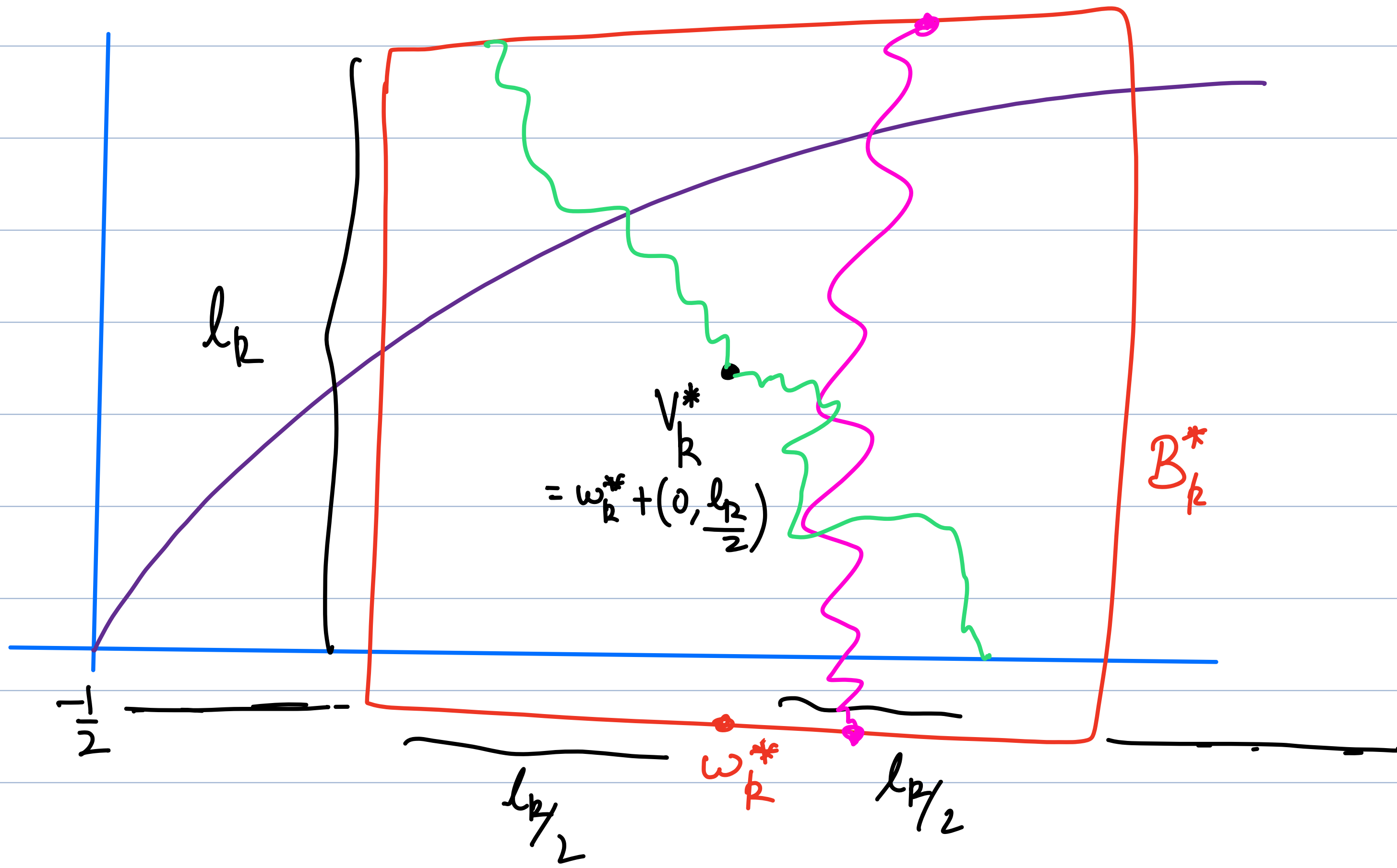
Let $w_k^* := \left(\lfloor k^{1+\delta} \rfloor + \frac{1}{2}, -\frac{1}{2}\right)$

Let B_k^* = Square box in the dual lattice such that

1) w_k^* is the midpoint of the bottomside of B_k^*

2) The height l_k of the box is s.t. it is the smallest box to contain the curve

$$\left\{ (x,y) : y = f(x),\ x \in \left[k^{1+\delta} + \frac{1}{2} - \frac{l_k}{2},\ k^{1+\delta} + \frac{1}{2} + \frac{l_k}{2}\right] \right\}$$



We know, $\frac{f(x)}{\log x} \longrightarrow a$ as $x \longrightarrow \infty$

$$l_k = a(1 + o(1)) \log k^{1+\delta} \quad \text{as } k \longrightarrow \infty$$

Let $A_k^* := \left\{ \exists \text{ a closed path lying in the dual box } B_k^*, \text{ connecting top and bottom of } B_k^* \right\}$

$$\supseteq \left\{ \exists \text{ two closed paths in } B_k^* \text{ - one connecting } v_k^* \text{ to the top of the box and other to the bottom of } B_k^* \right\}$$

By FKG,

$$\mathbb{P}(A_k^*) \geq \mathbb{P}_p(\bullet) \underset{(FKG)}{\geq} \mathbb{P}_p\left(v_k^* \xleftrightarrow{\text{closed}} \text{top of } B_k^* \text{ inside}\right) \cdot \mathbb{P}_p\left(v_k^* \xleftrightarrow{\text{closed}} \text{bottom of } B_k^* \text{ inside}\right)$$

$$= \left(\mathbb{P}_p\left(v_k^* \xleftrightarrow{\text{closed}} \text{top of } B_k^* \text{ inside}\right)\right)^2$$

$$\underset{(\text{Square root trick})}{\geq} \left(\frac{1}{4}\mathbb{P}_p\left(v_k^* \xleftrightarrow{\text{closed}} B_k^* \text{ inside}\right)\right)^2$$

$$\underset{(\text{def}^n \text{ of dual})}{=} \left[\frac{1}{4}\mathbb{P}_{1-p}\left(0 \xleftrightarrow{\text{open}} \delta B_{\ell_k/2}\right)\right)^2$$

Now

$$\mathbb{P}_{1-p}\left(0 \xleftrightarrow{\text{open}} \delta B_{\ell_k/2}\right) \approx \exp\left(\frac{-\ell_k}{2\,\xi(1-p)}\right) \quad \text{as } k \to \infty$$

$$\mathbb{P}(A_k^*) \geq \frac{1}{16} \exp\left(\frac{-2a \log k^{1+\delta} (1+o(1))}{2\xi(1-p)} \right)$$

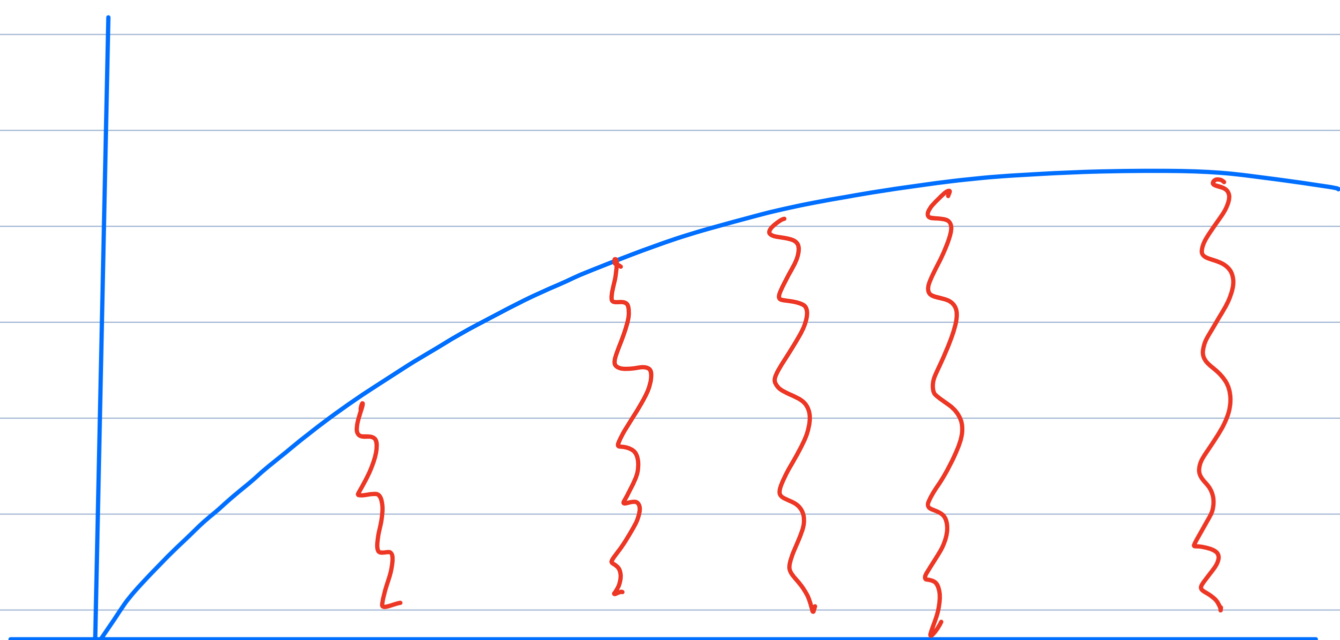
as $k \to \infty$

$$= \frac{1}{16} \exp\left(- \frac{a(1+\delta)(1+o(1)) \log k}{\xi(1-p)} \right)$$

$$= \frac{1}{16} k^{-\left((1+o(1))(1+\delta)a/\xi(1-p)\right)}$$

as $k \longrightarrow \infty$

$\sum_k \mathbb{P}(A_k^*) = \infty$, Suppose we can make the boxes disjoint then we got infinitely often red closed paths as below which make the existence of an infinite cluster impossible.



Claim: $\exists$ K s.t. $\{A_k^* : k \geq K\}$ are independent.

$\bullet\, \omega_k^*$ $\longleftrightarrow$ ω_{k+1}^*

$k^{1+\delta}$ $= (k+1)^{1+\delta}$

$k^{1+\delta} - k^{\delta} \sim (1+\delta)\, k^{\delta}$, while $l_k \sim O(\log k)$

So $\exists$ K s.t. the boxes are disjoint for some $k \geq K$

Now suppose that $p > \frac{1}{2}$ and $a > \xi(1-p)$. We'll show that $\exists$ a unbdd open component w.p. 1 in G_f.

Get α s.t. $a > \alpha > \xi(1-p)$

Let B_k^* = Square box in the dual lattice centered at $(k+\frac{1}{2}, \frac{1}{2})$ with side $2\alpha \log k$

For all k large enough B_k^* lies below the curve $y = f(x) - 2$

$$D_k^* = \left\{ \left(k+\tfrac{1}{2}, \tfrac{1}{2}\right) \xleftrightarrow{\text{closed}} \delta B_k^* \text{ in the dual lattice} \right\}$$

$$\mathbb{P}_p(D_k^*) = \mathbb{P}_{1-p}\left(0 \xleftrightarrow{\text{Closed}} \delta B_{\alpha \log k} \text{ in } \mathbb{Z}^2\right)$$

$$\approx k^{-\frac{\alpha}{\xi}(1-p)} \qquad (\text{as before})$$

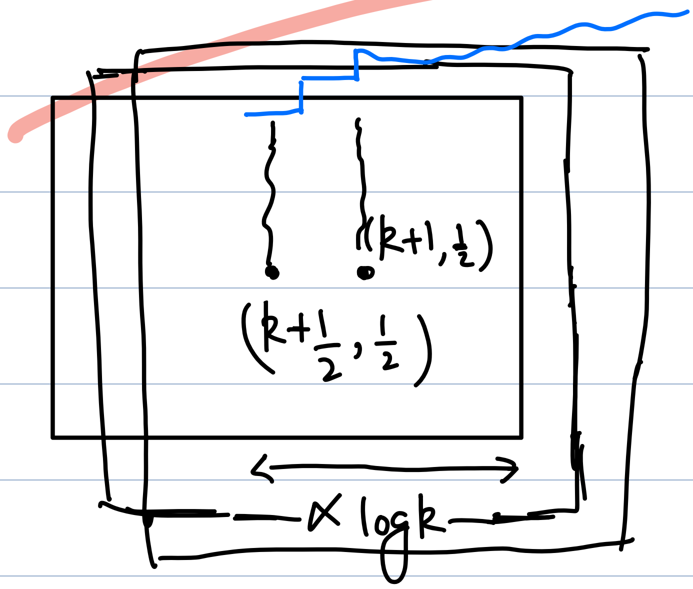
$$\therefore \sum \mathbb{P}_p(D_k^*) < \infty$$

So $\exists$ $M \geq 1$ s.t.

$$\mathbb{P}_p\Big(\bigcup_{k \geq M} D_k^*\Big) < \frac{1}{2}$$

or $\mathbb{P}_p\Big(\bigcap_{k \geq M} (D_k^*)^c\Big) > \frac{1}{2}$

For all $k \geq M$ w.p. $> \frac{1}{2}$ the dual closed path starting from $\left(k+\frac{1}{2}, \frac{1}{2}\right)$ stops short of the top boundary.



$$\mathbb{P}_p\left(G_f \text{ admits an unbdd open component}\right) > \frac{1}{2}$$

$$= 1$$

Thus $\xi(1-p_c) = a$

■

<u>What happens at</u> p_c?

$a>0$, $f(x) = a\log x + b\log(\log x)$

for x large

$b > 2a$

<u>Note</u>: $\dfrac{f(x)}{\log x} \longrightarrow a$ as $x \longrightarrow \infty$

Let B_k^*, D_k^* be as in the second part of the proof.

B_k^* has width $2f(k) + O(1)$ as $k \to \infty$

$$\mathbb{P}_p(D_k^*) = \mathbb{P}_{1-p}\left(0 \xleftrightarrow{\text{open}} \delta B_{f(k)+O(1)}\right)$$

$$(*) \qquad \leq \sigma \log k \exp\left(-\frac{a\log k + b\log(\log k)}{\xi(1-p)}\right)$$

So if π is the solution of the equation $a = \xi(1-p)$ then from we have,

$$\mathbb{P}_{\pi}(D_k^*) \leq \frac{\sigma}{k(\log k)^{b/a - 1}}$$

Now $b/a - 1 > 1$, then $\sum_k \mathbb{P}(D_k^*) < \infty$

Then by the same argument above, we are done.

$\boxtimes$

<u>Next Class</u>: Comparing bond and site percolation.