Post Midsem

The two point connectivity function (on $\mathbb{Z}^d$)

For $u, v \in \mathbb{Z}^d$, define, $\tau_d(u,v) = \mathbb{P}_p(u \longleftrightarrow v)$

$$= \mathbb{P}_p(0 \longleftrightarrow v-u)$$ [by Transl invariance]

Notation: We say $a_n \approx b_n$ as $n \to \infty$ if $\frac{\log a_n}{\log b_n} \to 1$ as $n \to \infty$

We have already discussed that

$$c_1(n,d)\, e^{-n\phi(p)} \le \mathbb{P}_p(0 \longleftrightarrow \delta B_n) \le c_2(n,d)\, e^{-n\phi(p)}$$

We showed that,

$$\boxed{\mathbb{P}_p(0 \longleftrightarrow \delta B_n) \approx e^{-n\phi(p)}}$$

Shown using methods of reliability theory.

Thm: Take $0 < p \le 1$ and $\phi(p)$ as before. Let $e_n = (n, 0, 0 \cdots 0)$, then we have

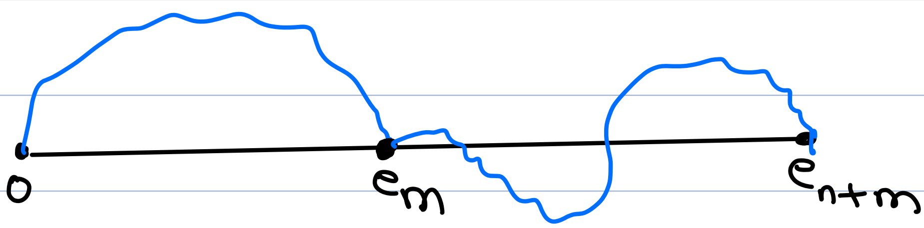
$$\lim_{n\to\infty}\left\{-\frac{1}{n}\log\left(\tau_p(0,e_n)\right)\right\} = \phi(p)$$

And, $\exists$ a positive constant $\xi > 0$ (not dependent on p) s.t.

$$\xi p\, n^{4(1-d)} e^{-n\phi(p)} \le \tau_p(0,e_n) \le e^{-n\phi(p)}$$

And hence, $\tau_p(0,e_n) \approx e^{-n\phi(p)}$

Proof : $\{0 \longleftrightarrow e_{n+m}\} \supseteq \{0 \longleftrightarrow e_m\} \cap \{e_m \longleftrightarrow e_{m+n}\}$



$\therefore$ Taking $t_k = -\log \tau_p(0,e_k)$, then we get

$$t_{m+n} \le t_m + t_n$$

Then by Fekete's Subadditive lemma,

$$\lim_{n\to\infty}\frac{t_n}{n} = \inf_{n\ge 1}\frac{t_n}{n}$$

Def^n: Let $\eta(p) := \lim_{n\to\infty} -\frac{1}{n}\log\left(\tau_p(0,e_n)\right)$

$$= \inf_{n \geq 1} \left\{ -\frac{\log(\tau_p(0, e_n))}{n} \right\}$$

<u>Note</u>: $\log(\tau_p(0, e_n)) \leq -n\,\eta(p) \quad \forall\, n \geq 1$

Now we show, $\eta(p) = \phi(p)$

Obviously, $\{0 \longleftrightarrow e_n\} \subseteq \{0 \longleftrightarrow \delta B_n\} \Rightarrow \eta(p) \geq \phi(p)$.

$$\left[\because\ \frac{-\log(\mathbb{P}_p(0 \leftrightarrow \delta B_n))}{n} \leq \frac{-\log \mathbb{P}_p(0 \leftrightarrow e_n)}{n} \right]$$

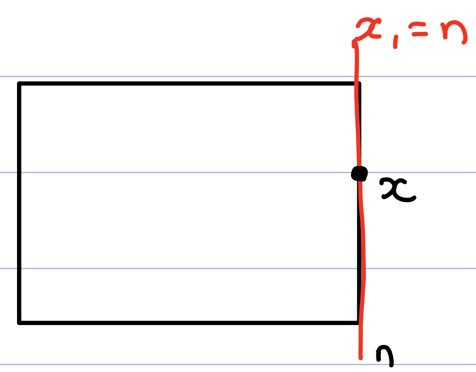
So, we need to show $\eta(p) \leq \phi(p)$.

A simple union bound tells us that,

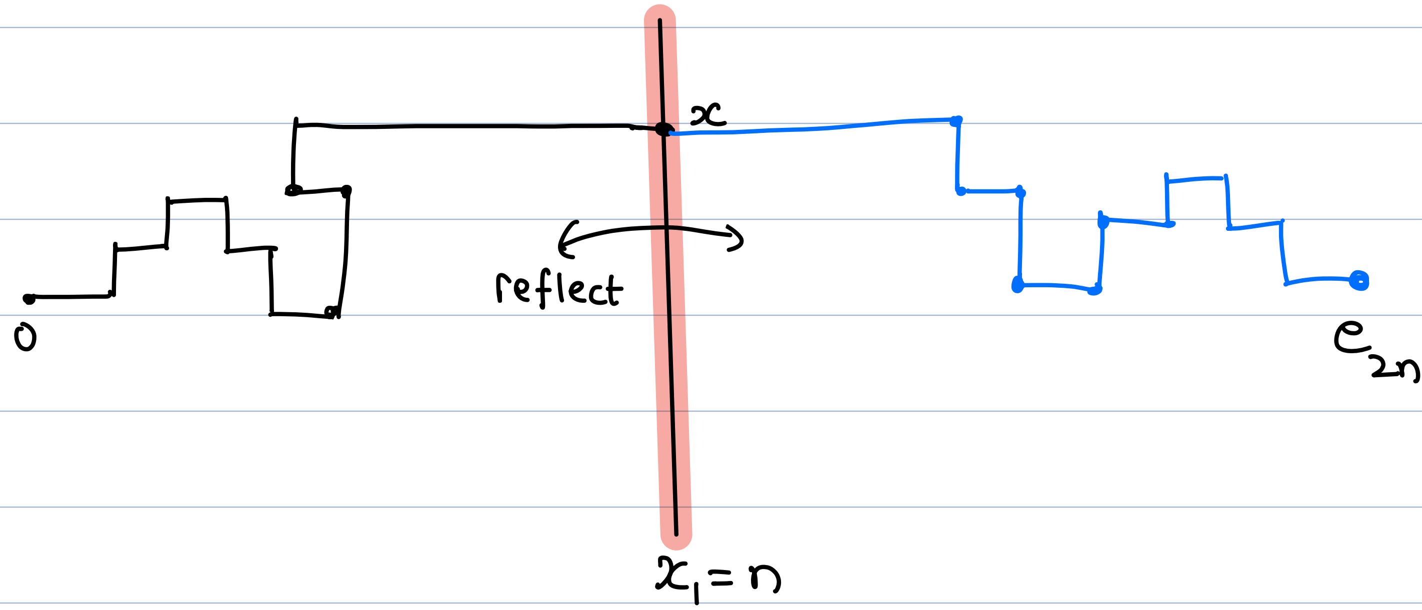
$$\mathbb{P}_p(0 \longleftrightarrow x) \geq \frac{1}{\# \delta B_n} \mathbb{P}_p(0 \longleftrightarrow \delta B_n) \quad \text{--- (1)}$$

for some $x \in \delta B_n$

By translation invariance, assume that the first coord of x is $x_1 = n$.



Now consider the reflected path, along the line $x_1 = n$. Then,



Clearly, $\{0 \leftrightarrow x\} \cap \{x \leftrightarrow e_{2n}\} \subseteq \{0 \leftrightarrow e_{2n}\}$

Thus, by FKG

$$\mathbb{P}_p(0 \leftrightarrow x)\,\mathbb{P}_p(x \leftrightarrow e_{2n}) \le \mathbb{P}_p(0 \leftrightarrow e_{2n})$$

Since $\mathbb{P}_p(0 \leftrightarrow x) = \mathbb{P}_p(x \leftrightarrow e_{2n})$ (See the figure above)

$$\left(\tau_p(0,x)\right)^2 \le \mathbb{P}_p(0 \leftrightarrow e_{2n})$$

$$\therefore \quad \tau_p(0,e_{2n}) \ge \frac{1}{\left(2d(2n)^{d-1}\right)^2}\left(\mathbb{P}_p(0 \leftrightarrow \delta B_n)\right)^2$$

(by (1))

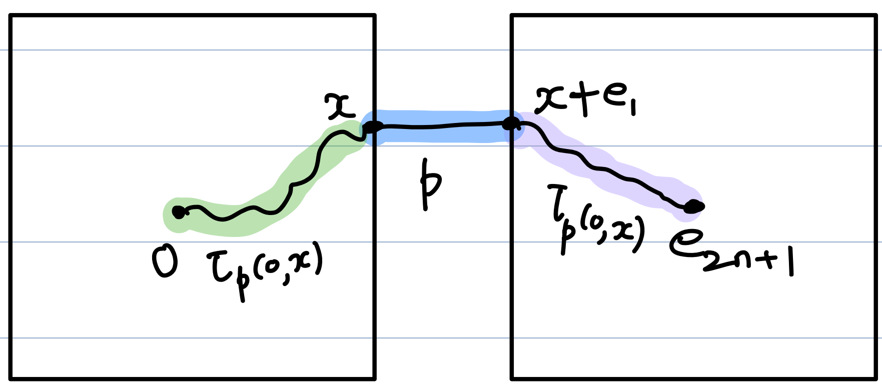
$$\tau_p(0,e_{2n}) \geq C n^{4(1-d)} \exp(-2n\phi(p))$$

$$\therefore \; -\frac{1}{2n} \tau_p(0,e_{2n}) \leq \log(\text{poly}) + \phi(p)$$

Now let's move to the odd case

Firstly,

$$\{0 \leftrightarrow x\} \cap \{x \leftrightarrow x+e_1\} \cap \{x+e_1 \leftrightarrow e_{2n+1}\}$$
$$\subseteq \{0 \leftrightarrow e_{2n+1}\}$$



$\Rightarrow$

$$\tau_p(0,e_{2n+1}) \geq p(\tau_p(0,x))^2$$

And hence by the same logic $-\frac{1}{2n+1}\log(\tau_p(0,e_{2n+1}))$

$$\leq o(n) + \phi(p)$$

And hence $\lim_{n\to\infty} \frac{-1}{n} \log(\tau_p(0,e_n)) \leq \phi(p)$

Thus, $\eta(p) = \phi(p)$

This also shows the theorem

$\boxtimes$

General Connectivity in $\mathbb{Z}^d$:

We are interested in $\tau_p(0,x)$, $x \in \mathbb{Z}^d$.

Prop^n (*): Let $p \in (0,1)$ and $\phi(p)$ as before. Then $\exists$ $\lambda > 0$ independent of p s.t.

$$\lambda p^d |x|^{4d(1-d)} e^{-|x|\phi(p)} \leq \tau_p(0,x) \leq \exp\left(-\|x\|\phi(p)\right)$$

Where, $|x| = \sum_{i=1}^{d} |x_i| = d_{\mathbb{Z}^d}(0,x)$ (graph dist)

$\|x\| = \max\{|x_i| : i = 1, \dots d\}$ = length of the smallest box centered at 0 containing x

Proof:

① Upper bound

Let $\|x\| = n$, then by the same argument as above

$$\tau_p(0,x) \le \sqrt{\tau_p(0,e_{2n})} \le \exp(-n\phi(p))$$

② Lower bound

Say in $\mathbb{Z}^3$, we wish to have a path from 0 to $(x_1,-x_2,-x_3)$

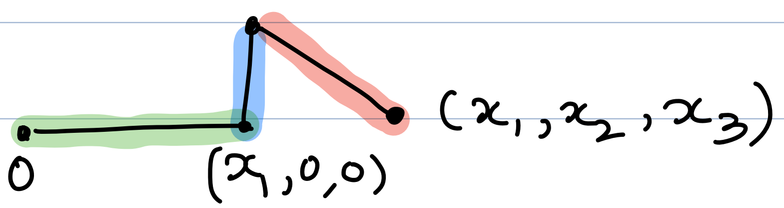
This has the same prob as having paths from 0 to $(x_1,x_2,-x_3)$ or to (x_1,x_2,x_3)

[reflection principle]

For $x \in \mathbb{Z}^d$ with $|x| = n$ and $\tilde{x} = (|x_1|, \cdots |x_d|)$ then $|\tilde{x}| = n$, also from the above observation $\tau_p(0,x) = \tau_p(0,\tilde{x})$. So WLOG, let $x_i \ge 0 \ \forall i$

Let $x^{(k)} = \{|x_1|, \cdots, |x_k|, 0 \cdots 0\}$ for $1 \le k \le d$

Then, $\{0 \leftrightarrow \tilde{x}\} \supseteq \{0 \leftrightarrow x^{(1)}\} \cap \{x^{(1)} \leftrightarrow x^{(2)}\} \cap \cdots \{x^{(d-1)} \leftrightarrow x^{(d)}\}$



$\therefore$ By FKG inequality,

$$\tau_p(0,x) \geq \prod_{i=1}^{d} \tau_p(x^{(i-1)}, x^{(i)})$$
$$= \prod_{i=1}^{d} \tau_p(0, (0, \cdots |x_i|, 0 \cdots 0))$$
$$= \prod_{i=1}^{d} \tau_p(0, e_{|x_i|}) \quad \left(\approx e^{-|x|\phi(p)}\right)$$

$$\geq \prod_{i=1} c\, p\, |x_i|^{4(1-d)} e^{-|x_i|\phi(p)}$$

(By previous result)

$$\therefore \tau_p(0,x) \geq C p^d \left(|x_1| \cdots |x_d|\right)^{4(1-d)} e^{-|x|\phi(p)}$$

Since $1-d<0$, by AM-GM inequality,

$$\tau_p(0,x) \geq \lambda p^d |x|^{4d(1-d)} e^{-|x|\phi(p)}$$

$\blacksquare$

Why "Correlation length"?

At $p=p_c$, we know that $\phi(p)=0$. At criticality, the physcists have a guess :

$\tau_{p_c}(0, e_n) \sim n^{2-d-\eta}$ for some $\eta > 0$

- At Criticality $\longleftarrow$ Polynomial decay
- Below Criticality $\longleftarrow$ exponential decay

When can we distinguish between a polynomial and a exponential decay?

Ans: When $n\phi(p)$ is large

$\xi(p) = \frac{1}{\phi(p)}$ — Correlation length

$\mathbb{P}_p(0 \longleftrightarrow \delta B_n) \approx e^{-\frac{n}{\xi(p)}}$ The exponential term becomes significant only when $n \gg \xi(p)$

$\tau_p(0 \longleftrightarrow e_n) \approx e^{-\frac{n}{\xi(p)}}$ — Gives a notion of lengthscale at which these behaviours can be distinguished

natural length scale where for distances smaller than $\xi(p)$, sites are likely to be correlated, for sites at a dist $> \xi(p)$ they are effictively uncorrelated.

Recall,

$$\theta(p) = \mathbb{P}_p(\# C(0) = \infty)$$

$$\chi(p) = \mathbb{E}_p(\# C(0)) \; ; \; p_T := \inf\{p : \chi(p) = \infty\}$$

Exc: Show $p_T = p_c$ (was open for > 40 years)

Prop^n: For $p < p_c$, $\tau_p(0, x) \leq \left(1 - \frac{1}{\chi(p)}\right)^{|x|}$

$\forall x \in \mathbb{Z}^d$

Proof:

Let $S_n := \{v \in \mathbb{Z}^d : |v| \leq n\}$ — ball in graph metric

$\partial S_n = \{v \in \mathbb{Z}^d : |v| = n\}$

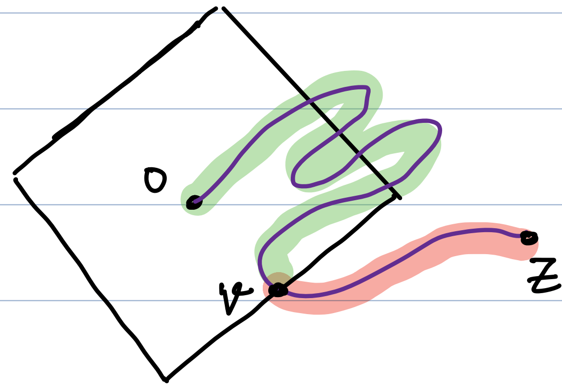
Let $M_n = \#(C \cap \partial S_n)$ = no. of vertices on the boundary conn to the origin

$$\therefore \mathbb{E}_p(M_n) = \mathbb{E}_p\left(\sum_{v \in \partial S_n} \mathbb{1}_{\{0 \leftrightarrow v\}}\right)$$

$$\sum_{n=0}^{\infty} \mathbb{E}_p(M_n) = \sum_{n=0}^{\infty} \sum_{v \in \partial S_n} \tau_p(0 \leftrightarrow v)$$

$$= \sum_{w \in \mathbb{Z}^d} \tau_p(0, w) = \mathbb{E}_p(\# C(0)) = \chi(p)$$

Let $z \in S_m^c = \{v \in \mathbb{Z}^d : |v| > m\}$. For a path from $0 \longleftrightarrow z$, let v be the last vertex in S_m



$$\therefore \tau_p(0,z) = \mathbb{P}_p\Big(\bigcup_{v\in\partial S_m} \{0\longleftrightarrow v\}\,\square\,\{v\longleftrightarrow z\}\Big)$$

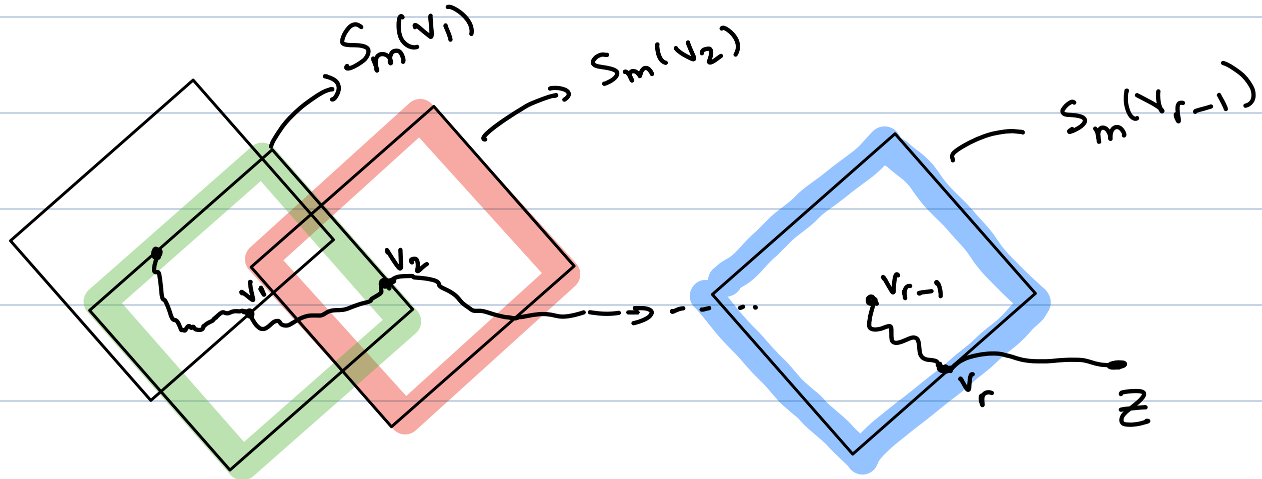
$$\underset{\substack{(BK+\text{union}\\ \text{bd})}}{\le} \sum_{v\in\partial S_m} \mathbb{P}_p(0\longleftrightarrow v)\,\mathbb{P}_p(v\longleftrightarrow z)$$

(∂B_n looks like a diamond)

$$= \sum_{v\in\partial S_m} \tau_p(0,v)\,\tau_p(v,z)$$

$$\le \sum_{v\in\partial S_m} \tau_p(0,v) = \mathbb{E}_p(N_m) \qquad \text{—} \quad (2)$$

Suppose $|z| = rm + s$, $0 \le s \le m$. Then



$$\tau_p(0,z) \le \sum_{\{v_1\cdots v_r\}} \mathbb{P}_p(0\longleftrightarrow v_1)\cdots \mathbb{P}_p(v_{r-1}\longleftrightarrow v_r)\,\mathbb{P}_p(v_r\longleftrightarrow z)$$

(BK multiple times)

$$\le \sum_{\{v_1 \cdots v_r\}} \mathbb{P}_p(0 \longleftrightarrow v_1) \cdots \mathbb{P}_p(v_{r-1} \longleftrightarrow v_r)$$

$$= \left(\mathbb{E}_p(M_m)\right)^r \quad — \quad (3)$$

For $u \in \mathbb{Z}^d$, $k \geq 1$,

$$\tau_p(0, ku) \geq \mathbb{P}_p\left(ju \longleftrightarrow (j+1)u \ \ \forall\, 0 \le j \le k-1\right)$$

$$\geq \left(\tau_p(0,u)\right)^k \qquad \left(\text{By FKG and translation inv}\right)$$

$$\therefore \ \tau_p(0,u) \le \left(\tau_p(0,ku)\right)^{\frac{1}{k}}$$

$$\le \left(\mathbb{E}_p(M_m)\right)^{\left\lfloor \frac{|ku|}{m} \right\rfloor \frac{1}{k}} \qquad \left(\text{By prev argument}\right)$$

$$\longrightarrow \left(\mathbb{E}_p(M_m)\right)^{\left\lfloor \frac{|u|}{m} \right\rfloor} \quad — \quad (4)$$

as $k \to \infty$

Finally for $0 < p < p_c = p_T$, $\chi(p) < \infty$

We'll show $\exists\, m$ s.t. $\mathbb{E}_p(M_m) \le \left(1 - \frac{1}{\chi(p)}\right)^m$ (+)

Assuming the existence of m as in $(\ddagger)$, using that m in (4) we get,

$$\tau_p(0,u) \leq \left(1-\frac{1}{\chi(p)}\right)^{\lfloor |u| \rfloor} \leq \left(1-\frac{1}{\chi(p)}\right)^{|u|}$$

$(\ddagger)$ is pretty easy to show since, if $\forall\ m$ $\mathbb{E}_p(M_m) > \left(1-\frac{1}{\chi(p)}\right)^m$, then using (2)

$$\chi(p) = \sum_{m \geq 1} \mathbb{E}_p(M_m) > \sum_m \left(1-\frac{1}{\chi(p)}\right)^m = \chi(p)$$

$$\Rightarrow \Leftarrow$$

So $(\ddagger)$ holds and hence our prop^n $\blacksquare$

So what did we achieve,

$$\phi(p) = \lim_n \left(-\frac{1}{n} \log \tau_p(0,e_n)\right)$$

(first thing we showed)

$$\underset{\text{(Prop}^n\text{ above)}}{\geq} \lim_n \left(-\frac{1}{n} \log\left[\left(1-\frac{1}{\chi(p)}\right)^n\right]\right)$$

$$\geq \lim_{n\to\infty} \left(-\frac{1}{n} \log\left(e^{-n/\chi(p)}\right) \right)$$

$$\left[1-x \geq e^{-x} \quad \forall x \right]$$

$$= \lim_{n\to\infty} \left(-\frac{1}{n} \cdot \frac{-n}{\chi(p)} \right) = \frac{1}{\chi(p)}$$

And hence $\quad \phi(p) \geq \dfrac{1}{\chi(p)} \Rightarrow \boxed{\xi(p) \geq \chi(p)}$