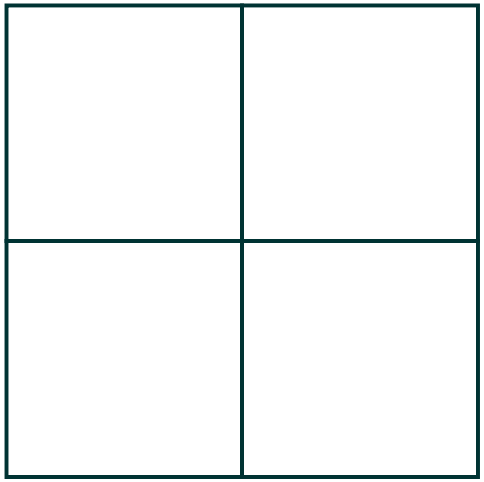
The OSSS inequality and its consequences in percolation

Rahul Roy
Indian Statistical Institute, New Delhi.

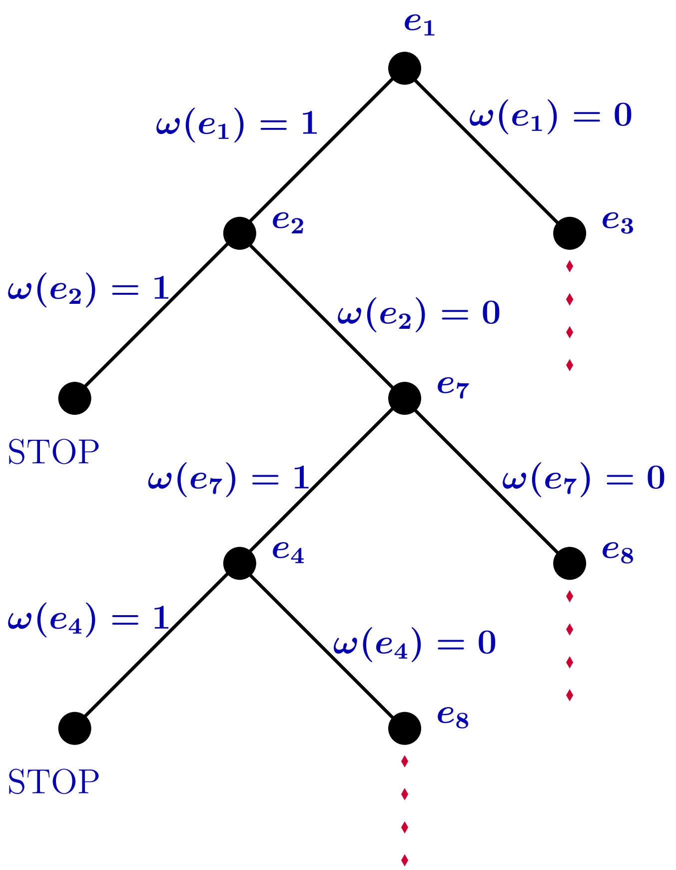
OSSS setup

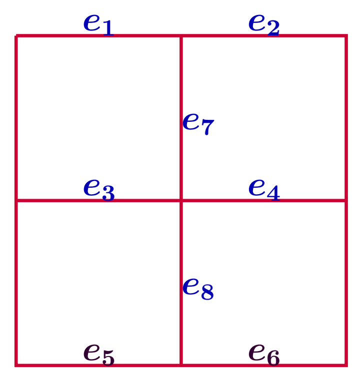
So we need to obtain the Duminil-Copin inequality, and it is here we need the OSSS inequality.

Let $\{(\Omega_i, \mathcal{F}_i, \mu_i) : 1 \leq i \leq n\}$ be a finite collection of measure spaces and $(\Omega, \mathcal{F}, \mu)$ be the product space.
Suppose we want to find whether there is a Left-Right open crossing of the following rectangle.



Decision tree





$A :=$ {∃ a L-R open crossing of the box}

$f := 1_A$

$f : \{0,1\}^8 \to [0,1]$

Let $\{(\Omega_i := \{0,1\}, \mathcal{F}_i, \mu_i) : 1 \leq i \leq n\}$ and $(\Omega, \mathcal{F}, \mu)$ be the product probability space.

Let $f : \Omega \to [0,1]$ be a measurable function (random variable) and fix a configuration ω. We want an algorithm T (decision tree) which opens (samples), one at a time, an $\omega(e_i)$ to determine the function.

We start with a root index i_1 and a family of decision rules $(\phi_j) : 1 \leq j \leq n-1$. The index i_{j+1} is chosen as

$$i_{j+1} := \phi_j(i_1, \ldots, i_j; \omega(e_{i_1}), \ldots \omega(e_{i_j})).$$

In our example, we chose $i_1 = 1$ and

$$i_2 = \phi_1(i_1, \omega(e_{i_1})) = \begin{cases} 2 & \text{if } \omega(e_1) = 1 \\ 3 & \text{if } \omega(e_1) = 0. \end{cases}$$

The algorithm T stops at step $\boldsymbol{\tau}$ if the value of the function is determined at time $\boldsymbol{\tau}$, i.e., $\mathrm{i}_1, \ldots, \mathrm{i}_{\boldsymbol{\tau}}$ and $\boldsymbol{\omega}(\mathrm{e}_{\mathrm{i}_1}), \ldots, \boldsymbol{\omega}(\mathrm{e}_{\mathrm{i}_{\boldsymbol{\tau}}})$ determine the function f.

Of course, $\boldsymbol{\tau}$ depends on the choice of the root index and the family of decision rules $(\phi_{\mathrm{j}}) : 1 \leq \mathrm{j} \leq \mathrm{n} - 1$.

In our example, if we observe

$$\mathrm{i}_1 = 1, \boldsymbol{\omega}(\mathrm{e}_{\mathrm{i}_1})) = 1; \qquad \mathrm{i}_2 = 2, \boldsymbol{\omega}(\mathrm{e}_{\mathrm{i}_2})) = 0;$$
$$\mathrm{i}_3 = 7, \boldsymbol{\omega}(\mathrm{e}_{\mathrm{i}_3})) = 1; \qquad \mathrm{i}_4 = 4, \boldsymbol{\omega}(\mathrm{e}_{\mathrm{i}_4})) = 1$$

then we stop at the 4th step, so $\boldsymbol{\tau} = 4$.

Definition

The algorithm T reveals edge e_j if $j \in \{i_1, \ldots, i_\tau\}$

Clearly the event $\{T \text{ reveals } e_j\}$ depends on the configuration $\omega(e_{i_1}), \ldots$ and we define the revealment of j as

$$\delta_{e_j}(T) := \mu\{T \text{ reveals } e_j\}.$$

We also define the influence of an edge e_i as

$$\mathrm{Inf}_i e_j := \mu\{f(\omega) \neq f(\omega')\} \text{ where } \omega'(e_j) = \begin{cases} \omega(e_j) & \text{if } j \neq i \\ 1 - \omega(e_i) & \text{if } j = i. \end{cases}$$

The OSSS inequality

The first result we have is useful for bond/site percolation on the lattice, later we state and prove the result useful for continuum models.

Theorem

(OSSS Inequality) Let f and $(\Omega, \mathcal{F}, \mu)$ be as above. In addition suppose that f is an increasing function. Then we have

$$\mathrm{Var}(f) \leq 2 \sum_{j=1}^{n} \delta_{e_j}(T)\,\mathrm{Cov}(\omega_{e_j}, f).$$

R. O'Donnell, M. Saks, O. Schramm and R. Servedio (2005)

Proof

First note that because f takes values in $[0, 1]$, we have

$$\mathrm{Var}(\mathrm{f}) = \mathrm{E}\left((\mathrm{f} - \mathrm{E}(\mathrm{f}))^2\right) \leq \mathrm{E}\left(|\mathrm{f} - \mathrm{E}\mathrm{f}|\right).$$

Now let $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ be two i.i.d. realizations from $\boldsymbol{\Omega}$.

$$\mathrm{E}(\mathrm{f} - \mathrm{E}(\mathrm{f})) = \mathbb{E}\left(\mathrm{f}(\boldsymbol{\omega}) - \mathrm{f}(\boldsymbol{\omega}')\right),$$

where $\mathbb{E}$ is w.r.t. the product measure $\boldsymbol{\mu} \times \boldsymbol{\mu}$. And so

$$\mathrm{Var}(\mathrm{f}) \leq \mathrm{E}(|\mathrm{f} - \mathrm{E}(\mathrm{f})|) \leq \mathbb{E}\left(|\mathrm{f}(\boldsymbol{\omega}) - \mathrm{f}(\boldsymbol{\omega}')|\right). \tag{1}$$

For the realisation $\boldsymbol{\omega}$ suppose the algorithm checks the edges $e_{i_1}, \ldots, e_{i_{\boldsymbol{\tau}}}$ before stopping.
Note that $\boldsymbol{\tau}$, as well as, the order of the edges $e_{i_1}, e_{i_2}, \ldots$ depend on the realisation $\boldsymbol{\omega}$. Let

$$J_t := \{t+1, t+2, \ldots, \boldsymbol{\tau}\}$$

and, define the configuration $\boldsymbol{\omega}^t$, as

$$\boldsymbol{\omega}^t(e_{i_j}) := \begin{cases} \boldsymbol{\omega}(e_{i_j}) & \text{for } j \in J_t \\ \boldsymbol{\omega}'(e_{i_j}) & \text{for } j \notin J_t \end{cases}$$

Clearly $f(\boldsymbol{\omega}^{\boldsymbol{\tau}}) = f(\boldsymbol{\omega}')$ and, the stopping time $\boldsymbol{\tau}$ ensures that $f(\boldsymbol{\omega}^0) = f(\boldsymbol{\omega})$, so that from (1) we have

$$
\begin{aligned}
\mathrm{Var}(f) &\leq E(|f - E(f)|) \\
&= \mathbb{E}\left(|f(\boldsymbol{\omega}) - f(\boldsymbol{\omega}')|\right) \\
&= \mathbb{E}\left(\sum_{t=1}^{\boldsymbol{\tau}} |f(\boldsymbol{\omega}^{t-1}) - f(\boldsymbol{\omega}^{t})|\right) \\
&\leq \sum_{t=1}^{n} \mathbb{E}\left(|f(\boldsymbol{\omega}^{t-1}) - f(\boldsymbol{\omega}^{t})| 1_{\{t \leq \boldsymbol{\tau}\}}\right) \\
&\leq \sum_{j=1}^{n} \sum_{t=1}^{n} \mathbb{E}\left(|f(\boldsymbol{\omega}^{t-1}) - f(\boldsymbol{\omega}^{t})| 1_{\{t = j\}}\right) \qquad (2)
\end{aligned}
$$

To study the term $f(\boldsymbol{\omega}^{t-1}) - f(\boldsymbol{\omega}^{t})$, we note that since f is increasing, on the event $\{i_t = j\}$ we have

$$
\begin{aligned}
&|f(\boldsymbol{\omega}^{t-1}) - f(\boldsymbol{\omega}^{t})| \\
&= \left(f(\boldsymbol{\omega}^{t-1}) - f(\boldsymbol{\omega}^{t})\right)\left(\boldsymbol{\omega}^{t-1}(e_j) - \boldsymbol{\omega}^{t}(e_j)\right) \\
&= f(\boldsymbol{\omega}^{t-1})\boldsymbol{\omega}^{t-1}(e_j) + f(\boldsymbol{\omega}^{t})\boldsymbol{\omega}^{t}(e_j) - f(\boldsymbol{\omega}^{t-1})\boldsymbol{\omega}^{t}(e_j) - f(\boldsymbol{\omega}^{t})\boldsymbol{\omega}^{t-1}(e_j)
\end{aligned} \tag{3}
$$

We will study each of the four terms above.

Let $X_t := (\boldsymbol{\omega}(e_{i_1}), \dots, \boldsymbol{\omega}(e_{i_{t\wedge\tau}}))$.

Observe that i_t is X_{t-1} measurable.

On the event $\{i_t = j\}$ we have

$$\begin{aligned} &\mathbb{E}\left(f(\boldsymbol{\omega}^{t-1})\boldsymbol{\omega}^{t-1}(e_j) \,\middle|\, X_{t-1}\right) \\ &= E\left(f(\boldsymbol{\omega})\boldsymbol{\omega}(e_j)\right) \text{ by independence} \\ &= \mathbb{E}\left(f(\boldsymbol{\omega}^{t})\boldsymbol{\omega}^{t}(e_j) \,\middle|\, X_t\right) \text{ by independence} \end{aligned} \tag{4}$$

Next, since f is increasing, so fixing the first few co-ordinates of $\boldsymbol{\omega}$, $f(\boldsymbol{\omega}^{t-1})\boldsymbol{\omega}^t(e_j)$ is increasing in $\boldsymbol{\omega}'$. So, by the FKG inequality,

$$\mathbb{E}\left(f(\boldsymbol{\omega}^{t-1})\boldsymbol{\omega}^t(e_j)\,\middle|\,X_n\right) \geq \mathbb{E}\left(f(\boldsymbol{\omega}^{t-1})\,\middle|\,X_n\right)\mathbb{E}\left(\boldsymbol{\omega}^t(e_j)\,\middle|\,X_n\right)$$

and

$$\mathbb{E}\left(\mathbb{E}\left(f(\boldsymbol{\omega}^{t-1})\boldsymbol{\omega}^t(e_j) \mid X_n\right)\,\middle|\,X_{t-1}\right)$$

$$\geq \mathbb{E}\left(\mathbb{E}(\boldsymbol{\omega}^t(e_j) \mid X_n)(\mathbb{E}(f(\boldsymbol{\omega}^{t-1}) \mid X_n))\,\middle|\,X_{t-1}\right)$$

$$= \mathbb{E}\left(\mathbb{E}(\boldsymbol{\omega}^t(e_j) \mid X_{t-1})\left[\mathbb{E}(f(\boldsymbol{\omega}^{t-1} \mid X_n))\,\middle|\,X_{t-1}\right]\right)$$

because i_t is determined by $i_1, \ldots, i_{t-1}$ and so

$\mathbb{E}\left(\boldsymbol{\omega}^t(e_j)\,\middle|\,X_n\right)$ is X_{t-1} measurable

$$= \mathbb{E}\left(\mathbb{E}(\boldsymbol{\omega}^t(e_j) \mid X_{t-1})(\mathbb{E}(f(\boldsymbol{\omega}^{t-1}) \mid X_{t-1})\right)$$

$$= E(\boldsymbol{\omega}(e_j))E(f)$$

because $\boldsymbol{\omega}^t$ and $\boldsymbol{\omega}^{t-1}$ are independent of X_{t-1}

Thus we have

$$\mathbb{E}\left(f(\boldsymbol{\omega}^{t-1})\boldsymbol{\omega}^{t}(e_j) \;\middle|\; X_{t-1}\right) \geq E(\boldsymbol{\omega}(e_j))E(f) \tag{5}$$

Similarly, we obtain

$$\mathbb{E}\left(f(\boldsymbol{\omega}^{t})\boldsymbol{\omega}^{t-1}(e_j) \;\middle|\; X_{t}\right) \geq E(\boldsymbol{\omega}(e_j))E(f) \tag{6}$$

Now combining everything from (2) and (3) we have

$$
\begin{aligned}
&\mathrm{Var}(f)\\
&\leq \sum_{j=1}^{n}\sum_{t=1}^{n}\big(\mathbb{E}(f(\boldsymbol{\omega}^{t-1})\boldsymbol{\omega}^{t-1}(e_j)1_{\{t=j\}}) + \mathbb{E}(f(\boldsymbol{\omega}^{t})\boldsymbol{\omega}^{t}(e_j)1_{\{t=j\}})\\
&\qquad -\mathbb{E}(f(\boldsymbol{\omega}^{t-1})\boldsymbol{\omega}^{t}(e_j)1_{\{t=j\}}) - \mathbb{E}(f(\boldsymbol{\omega}^{t})\boldsymbol{\omega}^{t-1}(e_j)1_{\{t=j\}})\big)
\end{aligned}
$$

taking conditional expectations appropriately w.r.t. $X_{t-1}$and X_t and then unconditioning and using (5) and (6)

$$
\begin{aligned}
&\leq \sum_{j=1}^{n}\sum_{t=1}^{n}\mathbb{E}\left((2E(f(\boldsymbol{\omega})\boldsymbol{\omega}(e_j)) - 2Ef(\boldsymbol{\omega})E\boldsymbol{\omega}(e_j))1_{\{t=j\}}\right)\\
&= \sum_{j=1}^{n} 2\mathrm{Cov}(f,\boldsymbol{\omega}(e_j))\sum_{t=1}^{n}E(1_{\{t=j\}})\\
&= 2\sum_{j=1}^{n}\boldsymbol{\delta}_{e_j}(T)\mathrm{Cov}(f,\boldsymbol{\omega}(e_j))
\end{aligned}
$$

□

The algorithm

So now we have to find a 'good' algorithm T which gives a good bound on $\delta_{e_j}(T)$.
A natural algorithm would be to first enumerate the edges E in $B_n = [-n, n]^d$. Note that for the event $\{0 \leftrightarrow \partial B_n\}$, we do not need to include the edges of ∂B_n.
So take $V_0 = 0$, the origin, and $E_0 = \emptyset$.
Let $e = < 0, v >$ (say) be the edge adjacent to the origin which is 'earliest' in the enumeration of E.
Take

$$E_1 = \{e\} \text{ and } V_1 = \begin{cases} V_0 & \text{if } \omega(e) = 0 \\ V_0 \cup \{v\} & \text{if } \omega(e) = 1 \end{cases}$$

For the next step get the edge $\mathrm{f} = < \mathrm{x}, \mathrm{y} > \notin \mathrm{E}_1$ with $\mathrm{x} \in \mathrm{V}_1$ and $\mathrm{y} \notin \mathrm{V}_1$, which is 'earliest' in the enumeration of $\mathrm{E} \setminus \mathrm{E}_1$.
Take

$$\mathrm{E}_2 = \mathrm{E}_1 \cup \{\mathrm{f}\} \text{ and } \mathrm{V}_2 = \begin{cases} \mathrm{V}_1 & \text{if } \boldsymbol{\omega}(\mathrm{f}) = 0 \\ \mathrm{V}_1 \cup \{\mathrm{y}\} & \text{if } \boldsymbol{\omega}(\mathrm{f}) = 1 \end{cases}$$

Continue in this fashion until one of the following two happen for the first time.

$$\mathrm{V}_\mathrm{t} \cap \partial \mathrm{B}_\mathrm{n} \neq \emptyset$$

in this case, you have found the open path from $\{0\}$ to $\partial \mathrm{B}_\mathrm{n}$

$$\text{there is no edge in } \mathrm{E} \setminus \mathrm{E}_\mathrm{t} \text{ incident to } \mathrm{V}_\mathrm{t}$$

in this case, there is no open path from $\{0\}$ to $\partial \mathrm{B}_\mathrm{n}$
Here $\boldsymbol{\tau} = \mathrm{t}$
Unfortunately, this algorithm does not give a good bound on $\boldsymbol{\delta}_{\mathrm{e}_\mathrm{j}}(\mathrm{T})$.

Duminil-Copin's work is to get an 'averaging' algorithm which gives a better bound.

We want an algorithm to obtain all the open connected components of B_n which intersect the square $\delta(B_k)$ for each $k \in \{1, \ldots, n\}$.

The algorithm T we define below starts with

$$V_0 = \{v : v \in \delta(B_k)\} \text{ and } E_0 = \emptyset.$$

Having defined $V_0 \subseteq V_1 \subseteq \cdots \subseteq V_s$ and $E_0 \subseteq E_1 \subseteq \cdots \subseteq E_s$

(Step 1) if there exists an edge $e =< x, y >\notin E_s$ with $x \in V_s$ and $y \notin V_s$, choose the one 'earliest' in the enumeration of $E \setminus E_s$. With a slight abuse of notation let $e =< x, y >$ be this edge. The decision rule ϕ_t chooses this edge e and set

$$E_{s+1} = E_s \cup \{e\} \text{ and } V_{s+1} = \begin{cases} V_s & \text{if } \omega(e) = 0 \\ V_s \cup \{y\} & \text{if } \omega(e) = 1. \end{cases}$$

(Step 2) if no such edge exists then take $E_{s+1} = E_s \cup \{e\}$, where e is the 'earliest' in the enumeration of E, with $e \notin E_s$ and $V_{s+1} = V_s$.

We note that in the first step we are still exploring whether an edge belongs to the connected open component of $\boldsymbol{\delta}(B_k)$. When this step stops, we are in exactly one of two situations

Situation 1. we have found a connected open component admitting a path from the origin $\{0\}$ to $\boldsymbol{\delta}(B_n)$

Situation 2. (i) we have found closed edges surrounding the origin $\{0\}$ in B_k which does not allow any open path from the origin $\{0\}$ to $\boldsymbol{\delta}(B_k)$ or

(ii) we have found closed edges surrounding $\boldsymbol{\delta}(B_k)$ in $B_n \setminus B_k$ which does not allow any open path from the origin $\boldsymbol{\delta}(B_k)$ to $\boldsymbol{\delta}(B_n)$.

In either case our goal has been achieved.

More importantly note that the stop time $\boldsymbol{\tau}$ is smaller than the time when the first step stops. Also $\boldsymbol{\tau}$ may be strictly smaller because Situation 1 or 2 may have been obtained even before all the connected open components of $\boldsymbol{\delta}(B_k)$ are found.

Thus for any edge $e = < u, v > \in E$ we see that the revealment of e for the algorithm to study $1_{\{0 \leftrightarrow \partial B_n\}}$ is smaller than that for the above algorithm. Hence

$$\boldsymbol{\delta}_e(T) \leq P_p\{u \leftrightarrow \boldsymbol{\delta}(B_k)\} + P_p\{v \leftrightarrow \boldsymbol{\delta}(B_k)\}.$$

W.l.o.g. assume that $u \notin \partial B_n$ and $v \neq 0$.

Taking $d_u = \max\{u_1, \ldots, u_d\}$, where $u = (u_1, \ldots, u_d)$, we see that

$$\sum_{k=1}^{n} P_p\{u \leftrightarrow \delta(B_k)\} \leq \sum_{k=1}^{n} P_p\{u \leftrightarrow (u + \delta(B_{k-d_u}))\}$$
$$\leq 2 \sum_{k=0}^{n-1} \theta_k. \tag{7}$$

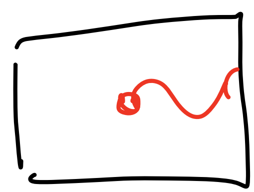
Combining everything we have

$$
\begin{aligned}
& n\boldsymbol{\theta}_n(p)(1-\boldsymbol{\theta}_n(p)) \\
& = n\mathrm{Var}(1_{\{0\leftrightarrow\partial B_n\}}) \text{ direct calculation} \\
& \leq 2\sum_{k=1}^{n}\sum_{e\in E}\boldsymbol{\delta}_e(T)\ \mathrm{Cov}(1_{\{0\leftrightarrow\partial B_n\}},\boldsymbol{\omega}(e)) \text{ OSSS} \\
& \leq 8\sum_{k=0}^{n-1}\boldsymbol{\theta}_k\sum_{e\in E}\ \mathrm{Cov}(1_{\{0\leftrightarrow\partial B_n\}},\boldsymbol{\omega}(e)) \text{ from (7)} \\
& = 8p(1-p)\boldsymbol{\theta}_n'(p)\sum_{k=0}^{n-1}\boldsymbol{\theta}_k \text{ Russo's formula} \\
& \leq 2\boldsymbol{\theta}_n'(p)\sum_{k=0}^{n-1}\boldsymbol{\theta}_k(p)
\end{aligned}
$$

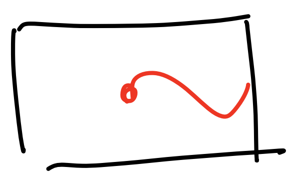
Finally, $\boldsymbol{\theta}_n(p)$ being a polynomial in p with $\boldsymbol{\theta}_p < 1$ for $p < 1$, we can get $\boldsymbol{\epsilon} > 0$ and a constant $c > 0$ such that $1 - \boldsymbol{\theta}(1 - \boldsymbol{\epsilon}) = c > 0$, which gives us

$$\boldsymbol{\theta}'_n(p) \geq c \frac{n\boldsymbol{\theta}_n(p)}{\sum_{k=0}^{n-1} \boldsymbol{\theta}_k(p)}$$

as required.

For $p < p_c$ $\quad \mathbb{P}_p\Big($  $\Big) \leq c_1 e^{-c_2 n}$

B_n

For $p > p_c$ $\quad \mathbb{P}_p\Big($  $\Big) \geq \frac{p - p_c}{2}$