Lecture 10

Last time: Duminil - Copin : $\theta_n'(p) \geq c \dfrac{n\,\theta_n(p)}{\sum_{k=0}^{n-1} \theta_k(p)}$

Lemma: Let $\{f_n\}_{n\geq 0}$ be a seq of inc and diff'ble functions satisfying

(i) $f_n : (a,b) \longrightarrow (0,M) \quad \forall\, n$

(ii) $\{f_n\}$ converges pointwise in (a,b)

(iii) $f_n' \geq \dfrac{n}{\sum_{k=0}^{n-1} f_k} f_n$

Then $\exists\ x_0 \in [a,b]$ s.t.

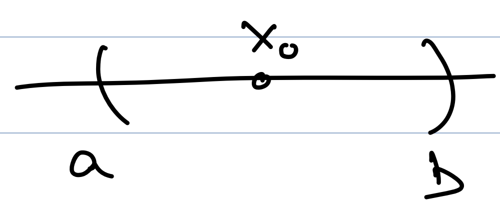
ⓐ $\forall\ x \in (a, x_0)$ and n large enough s.t.

$$f_n(x) \leq M \exp\left(-\frac{\sqrt{n}\,(x_0 - x)}{2}\right)$$

ⓑ $\forall\ x \in (x_0, b)$

$$f := \lim_{n\to\infty} f_n \quad \text{satisfies}$$

$$f(x) \geq \frac{x - x_0}{2}$$



Proof of lemma: (Analytic proof)

Let:

$$q := \min\left\{ b, \inf\left\{ x \in (a,b) : \limsup \frac{\log\left(\sum_{k=0}^{n-1} f_k(x)\right)}{\log n} \geq \frac{1}{2} \right\} \right\}$$

We'll show that the lemma holds for $q = x_0$

① For $q = a$, The lemma holds (See last part same argument)

② Suppose $q > a$

Take $x, y \in (a, q)$ s.t. $y = \frac{x+q}{2}$

$y < q$ so

$$\limsup_{n} \frac{\log\left(\sum_{k=0}^{n-1} f_k(y)\right)}{\log n} < \frac{1}{2}$$

So $\exists$ N s.t.

$$\frac{\log \sum_{k=0}^{n-1} f_k(y)}{\log n} < \frac{1}{2} \quad \forall\ n \geq N$$

So $\displaystyle\sum_{k=0}^{n-1} f_k(y) < \sqrt{n}$

For all $z \in [x,y]$; $\displaystyle\sum_{k=0}^{n-1} f_k(z) < \sqrt{n}$

By (iii) $f_n' \geq \sqrt{n}\, f_n(z)$

$$\int_x^y \frac{f_n'}{f_n(z)} \geq \int_x^y \sqrt{n} \Rightarrow \log f_n(y) - \log f_n(x) \geq \sqrt{n}\,(y-x)$$

$$\Rightarrow f_n(y) \geq f_n(x)\, e^{\sqrt{n}\,(y-x)}$$

or

$$f_n(x) \leq f_n(y)\, e^{-\sqrt{n}\,(y-x)}$$

$$f_n(x) \leq M\, e^{-\sqrt{n}\,(y-x)}$$
$$= M e^{-\sqrt{n}\left(\frac{q-x}{2}\right)}$$

This shows (a) for $q = x_0$

We need to show (b) holds for $x_0 = q$.

If $a = b$, then it's easy to see, so assume $a < b$.

Define T_n as $T_n(z) = \frac{1}{\log n} \sum_{i=1}^{n} \frac{f_i(z)}{i}$

Then
$$T_n'(z) = \frac{1}{\log n} \sum_{i=1}^{n} \frac{f_i'(z)}{i}$$
$$\geq \frac{1}{\log n} \sum_{i=1}^{n} \frac{f_i(z)}{\sum_{k=0}^{i-1} f_k(z)}$$

Now,
$$T_n'(z) \geq \frac{1}{\log n} \sum_{i=1}^{n} \left[\log \sum_{k=0}^{i} f_k(z) - \log \sum_{k=0}^{i-1} f_k(z) \right]$$

$\because$ For $0 < s < t$, $\frac{t-s}{s} = \int_s^t \frac{dr}{s} \geq \int_s^t \frac{dr}{r} = \log t - \log s$

$$= \frac{1}{\log n} \left[\log \sum_{k=0}^{n} f_k(z) - \log f_0(z) \right]$$
$$\geq \frac{1}{\log n} \left[\log \sum_{k=0}^{n} f_k(z) - \log M \right]$$

Take $a < w < z < b$

$$\frac{T_n(z) - T_n(w)}{z - w} = T_n'(\zeta) \quad \text{for some } \zeta \in (w, z)$$

$$\geq \frac{1}{\log n}\left[\log \sum_{k=0}^{n} f_k(\zeta) - \log M\right]$$

$$\geq \frac{1}{\log n}\left[\log \sum_{k=0}^{n} f_k(\omega) - \log M\right]$$

$\omega > q$ and f_k's are non-decreasing, so

$$\limsup_{n\to\infty} \frac{T_n(\zeta) - T_n(\omega)}{\zeta - \omega} \geq \frac{1}{2} \quad \text{(by choice of } q\text{)}$$

$\underline{\quad\quad}$ (*)

$\{T_n\}$ converges pointwise to f in (a,b)

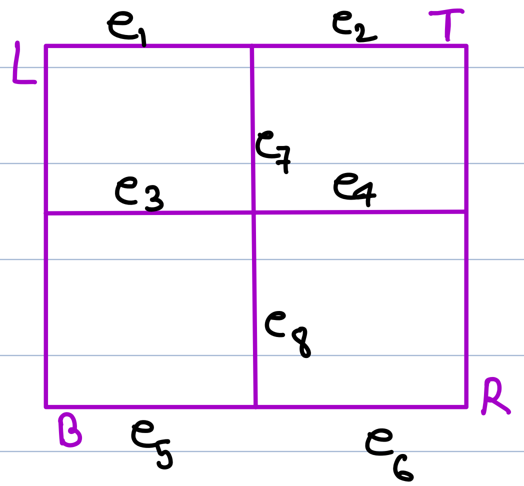
$\therefore \quad f(\zeta) \geq \frac{1}{2}(\zeta - \omega)$ for all $\omega > q$

$b > \zeta > q$

$\Rightarrow \quad f(\zeta) \geq \frac{1}{2}(\zeta - q) \quad \forall\, \zeta > q.$ ∎

OSSS Inequality

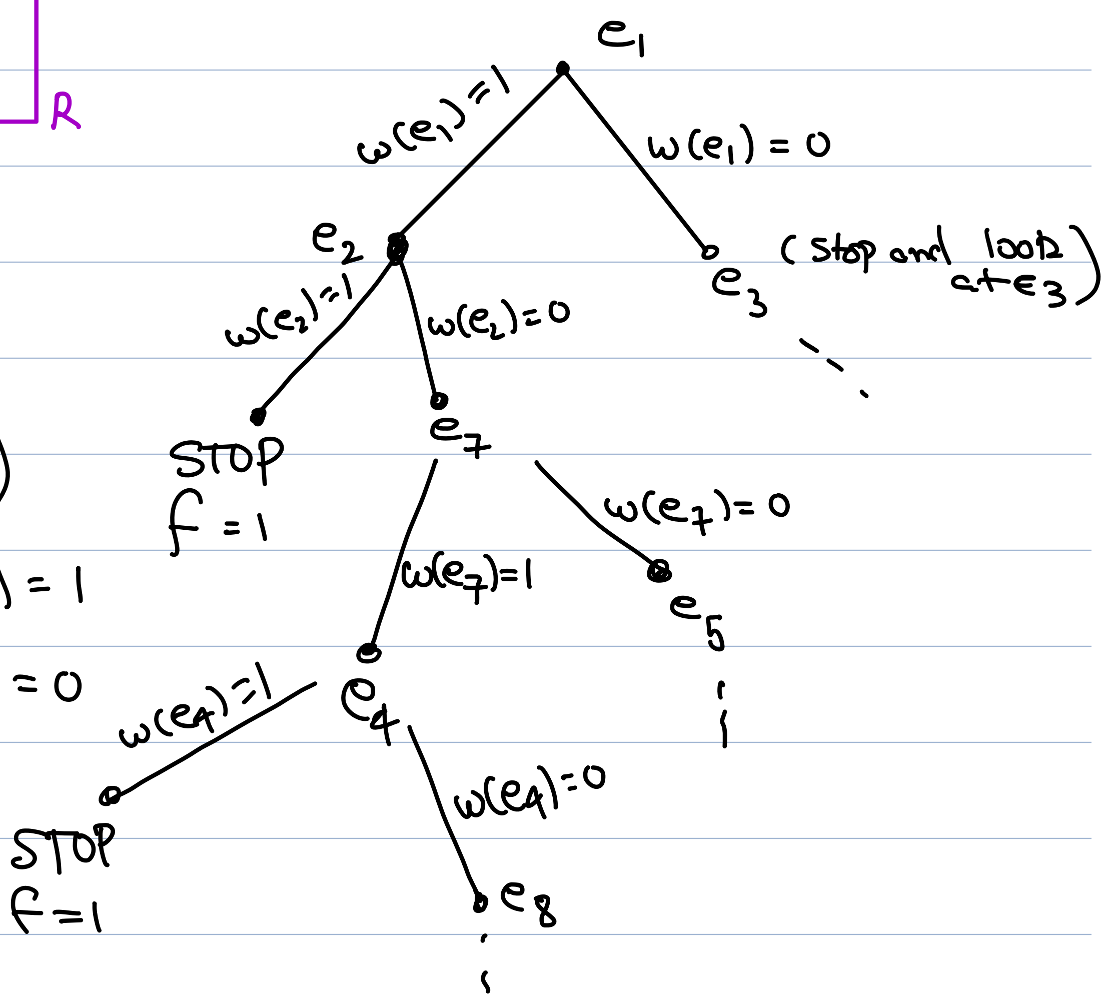
Let $\{(\Omega_i, \mathcal{F}_i, \mu_i) : 1 \leq i \leq n\}$ be a finite collection of measure spaces $(\Omega, \mathcal{F}, \mu)$ be the product space.



$A = \{ \exists$ a L-R open crossing of the box $\}$

$$\boxed{f = \mathbb{1}_A}$$

<u>Decision Tree</u> :



$i_1 = 1$

$$i_2 = \phi(i_1, \omega(e_1)) = \begin{cases} 2 & \text{if } \omega(e_1) = 1 \\ 3 & \text{if } \omega(e_1) = 0 \end{cases}$$

$$f : \{0,1\}^8 \longrightarrow [0,1]$$

<u>In general</u>,

$f : \Omega \to [0,1]$ measurable function and fix a configuration ω, we want an algorithm T (decision tree) which samples edges one at a time [an $\omega(e_i)$] to determine the func.

We started with a root index i_1 and a family of decision rules $\{\phi_j : 1 \le j \le n-1\}$

The index i_{j+1}

$$i_{j+1} = \phi_j \left(i_1, \dots i_j, w(e_{i_1}), \dots, w(e_{i_j}) \right)$$

The algorithm T stops at τ if the value of the function is determined at time τ, i.e. $i_1, i_2 \dots i_\tau, w_{i_1}, \dots, w_{i_\tau}$ determine the function f.

<u>Def^n</u>: The algorithm reveals edge e_j if

$$j \in \{i_1, \dots i_\tau\}$$

Def^n

$$\delta_{e_j}(T) := \mu\{T \text{ reveals } e_j\}$$ — revealment of j

$$\boxed{\text{We need independence (product measure) because we dont want } w(e_i) \text{ to influence } w(e_j)}$$

<u>Def^n</u>: The influence of an edge e_i

$$\mathrm{Inf}_i\, e_j = \mu\,(f(\omega) \neq f(\omega'))$$

where $\omega'(e_j) = \begin{cases} \omega(e_j) & \text{if } j \neq i \\ 1-\omega(e_i) & \text{if } j = \end{cases}$

Thm (OSSS inequality)

Let f, $(\Omega, \mathcal{F}, \mu)$ be as before, assume f is an increasing function, Then

$$\mathrm{Var}(f) \leq 2 \sum_{j=1}^{n} \delta_{e_j}(T)\, \mathrm{Cov}(\omega_{e_j}, T)$$