Exercise 7 *Prove* (1.7).

The equation 1.7 in the notes is

$$\mu^{\mathrm{f}}_{G,\beta}[\sigma_A] = \phi^0_{G,p,2}[\mathcal{F}_A] \quad — (1.7)$$

$G=(V,E)$

Sol : Recall the coupling in prop 1.2, i.e.

$$P((\omega,\sigma)) = \begin{cases} 0 & \text{if } (\omega,\sigma) \text{ are compatible} \\ \frac{1}{Z^0_{G,p,q}} p^{o(\omega)}(1-p)^{c(\omega)} & \text{o.w.} \end{cases}$$

$$\mu^f_{G,\beta}(\sigma_A) = \sum_{\omega\in\{0,1\}^E} \sum_{\sigma\in\{\pm1\}^V} \prod_{x\in A} \sigma_x \, P((\omega,\sigma))$$

$$= \sum_{\omega\in\{0,1\}^E} \sum_{\sigma\in\{\pm1\}^V} \left(\prod_{x\in A}\sigma_x\right)\left[\mathbb{1}(\omega\in\mathcal{F}_A)+\mathbb{1}(\omega\notin\mathcal{F}_A)\right] P((\omega,\sigma))$$

$$= \sum_{\omega\in\{0,1\}^E} P((\omega,\sigma))\left[\sum_{\sigma\in\{\pm1\}^V}\left(\prod_{x\in A}\sigma_x\right)\mathbb{1}(\omega\in\mathcal{F}_A) + \sum_{\sigma\in\{\pm1\}^V}\left(\prod_{x\in A}\sigma_A\right)\mathbb{1}(\omega\notin\mathcal{F}_A)\right]$$

$$= \sum_{\omega\in\{0,1\}^E} P((\omega,\sigma))\, \mathbb{1}(\omega\in\mathcal{F}_A) + 0$$

Since

$$\sum_{\sigma \in \{\pm 1\}^V} \prod_{x \in A} \sigma_x = \begin{cases} 1 & \text{if } \omega \in \mathcal{F}_A \\ 0 & \text{o.w.} \end{cases}$$

$$\therefore \quad \mu^{f}_{G,\beta}(\sigma_A) = \phi^{0}_{G,p,q}(\mathcal{F}_A)$$

This also shows the first Griffiths inequality i.e.

$$\mu^{f}_{G,\beta}(\sigma_A) \geq 0$$

$\blacksquare$