Exercise 23 (Zhang argument) *1. Show that*

$$\mathbb{P}_{1/2}[\text{top of } \Lambda_n \text{ is connected to infinity outside } \Lambda_n] \geq 1 - \mathbb{P}_{1/2}[\Lambda_n \not\leftrightarrow \infty]^{1/4}.$$

2. Deduce that the probability of the event $\mathcal{B}_n$ that there exist infinite paths in ω from the top and bottom of Λ_n to infinity in $\mathbb{Z}^2 \setminus \Lambda_n$, and infinite paths in ω^* from the left and right sides to infinity satisfies

$$\mathbb{P}_{1/2}[\mathcal{B}_n] \geq 1 - 4\mathbb{P}_{1/2}[\Lambda_n \not\leftrightarrow \infty]^{1/4}.$$

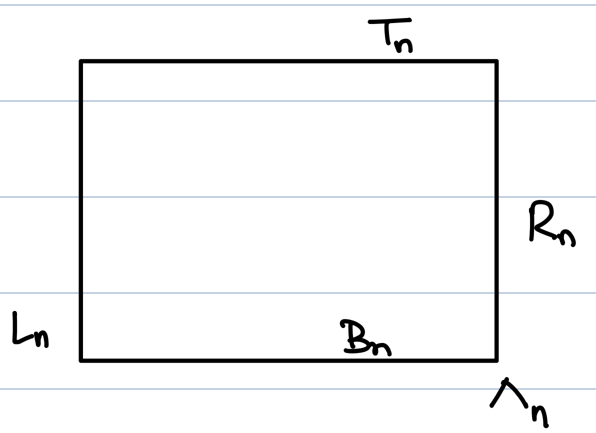
3. Using (FE) and the uniqueness of the infinite cluster, prove that $\mathbb{P}_{1/2}[\Lambda_n \not\leftrightarrow \infty]$ cannot tend to 0.

Sol :

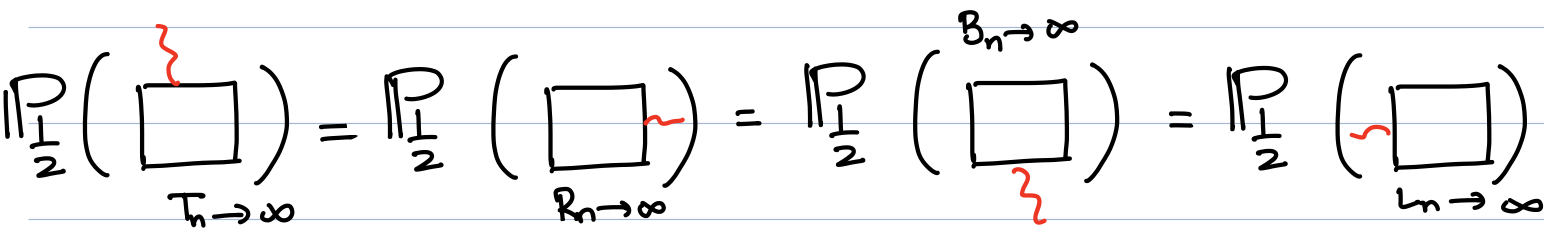
① To show

$$\mathbb{P}_{\frac{1}{2}}\left(\text{top of } \Lambda_n \text{ is connected to infinity outside } \Lambda_n\right) \geq 1 - \mathbb{P}_{\frac{1}{2}}\left(\Lambda_n \not\leftrightarrow \infty\right)^{\frac{1}{4}}$$

Proof:

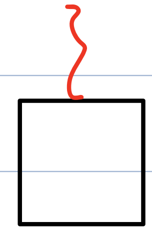


Let the top, bottom, right, left sides of Λ_n be T_n, B_n, R_n, L_n. By symmetry:



Also note that $\{ I_n \longrightarrow \infty \text{ outside of } \Lambda_n \}$ is increasing for $I \in \{T, R, B, L\}$.

Therefore by the squareroot trick

$$\mathbb{P}_{\frac{1}{2}}\Big($$  $$\Big) \geq 1 - \mathbb{P}_{\frac{1}{2}}\Bigg(\bigcap_{I \in \{T,B,R,L\}} \left\{ \begin{array}{l} I_n \text{ is not connected} \\ \text{to } \infty, \text{ outside} \\ \text{of } \Lambda_n \end{array} \right\} \Bigg)$$

$$= 1 - \mathbb{P}_{\frac{1}{2}}(\Lambda_n \not\longleftrightarrow \infty)$$

This shows ①

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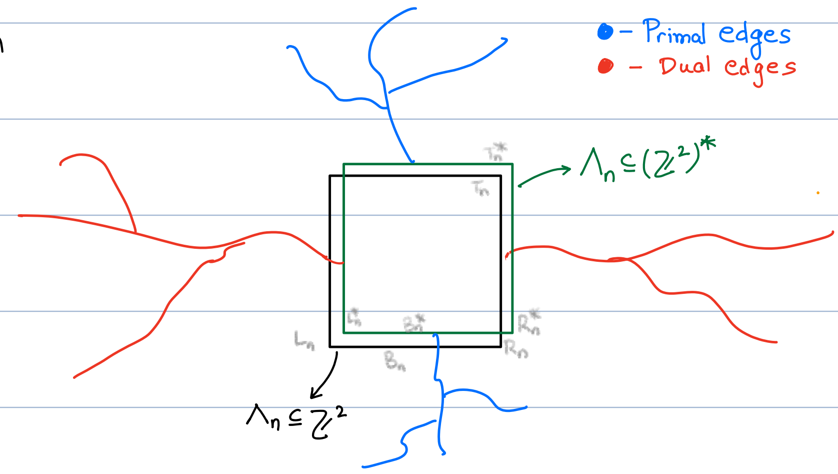
② $\mathcal{B}_n = \Bigg\{$ 

$\Bigg\}$

$$\mathbb{P}_{\frac{1}{2}}(\mathcal{B}_n) = \mathbb{P}_{\frac{1}{2}}\left(\begin{array}{l} \overbrace{\{T_n \to \infty \text{ in } \mathbb{Z}^2 \setminus \Lambda_n\}}^{\mathrm{I}} \cap \overbrace{\{B_n \to \infty \text{ in } \mathbb{Z}^2 \setminus \Lambda_n\}}^{\mathrm{II}} \\ \underbrace{\{R_n^* \to \infty \text{ in } (\mathbb{Z}^2)^* - (\Lambda_n)^*\}}_{\mathrm{III}} \cap \underbrace{\{L_n^* \to \infty \text{ in } (\mathbb{Z}^2)^* - (\Lambda_n)^*\}}_{\mathrm{IV}} \end{array} \right)$$

Let the above events be I, II, III, IV.

Then $\mathbb{P}(\mathcal{B}_n) = 1 - \mathbb{P}\left(\begin{array}{l} \text{at least one of } \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\} \\ \text{does not happen} \end{array} \right)$

Since we are working at $p = \frac{1}{2}$ the processes on $\mathbb{Z}^2$ and $(\mathbb{Z}^2)^*$ are exactly the same, therefore $\mathbb{P}_{\frac{1}{2}}(\mathrm{I}) = \mathbb{P}_{\frac{1}{2}}(\mathrm{II}) = \mathbb{P}_{\frac{1}{2}}(\mathrm{III}) = \mathbb{P}_{\frac{1}{2}}(\mathrm{IV})$

Now by (1), $\mathbb{P}_{\frac{1}{2}}(\mathrm{I} \text{ does not happen}) \leq \mathbb{P}(\Lambda_n \not\leftrightarrow \infty)^{\frac{1}{4}}$

Thus by union bound

$$\mathbb{P}(\mathcal{B}_n) \geq 1 - 4\,\mathbb{P}_{\frac{1}{2}}(\Lambda_n \not\leftrightarrow \infty)^{\frac{1}{4}}$$

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(3) Now Suppose $\mathbb{P}_{\frac{1}{2}}(\Lambda_n \not\leftrightarrow \infty)$ tends to 0 as $n \to \infty$.

Then $\mathbb{P}(\mathcal{B}_n) \longrightarrow 1$ as $n \to \infty$.

Let $\mathcal{A}_n$ be the event that all edges in Λ_n are open, Then by Finite Energy (FE)

Pick n s.t. $\mathbb{P}_{\frac{1}{2}}(\mathcal{A}_n \cap \mathcal{B}_n) > 0$, it is easy to see $\mathcal{A}_n \cap \mathcal{B}_n \subseteq \{\exists \text{ atleast } 2 \text{ infinite clusters in } \mathbb{Z}^2\}$

Thus $\mathbb{P}\left(\begin{array}{l}\exists \text{ atleast } 2 \text{ infinite} \\ \text{clusters}\end{array}\right) > 0$

$\Rightarrow \Leftarrow$

So $\mathbb{P}_{\frac{1}{2}}(\Lambda_n \not\leftrightarrow \infty)$ cannot tend to

0, and thus $p_c \geq \frac{1}{2}$.

$\blacksquare$