Exercise 22 (Square root trick) *Prove, using* (FKG), *that for any increasing events* $\mathcal{A}_1, \dots, \mathcal{A}_r$,

$$\max\{\mathbb{P}_p[\mathcal{A}_i] : 1 \le i \le r\} \ge 1 - \Big(1 - \mathbb{P}_p\Big[\bigcup_{i=1}^{r} \mathcal{A}_i\Big]\Big)^{1/r}.$$

Sol: The following Claim will imply the exc easily.

Claim: $\mathbb{P}(A_1^c \cap A_2^c \cap A_3^c \cdots \cap A_r^c) \geq \prod_{i=1}^{r} \mathbb{P}_p(A_i^c)$

We'll show this by induction, for $r = 1$ it's trivially true. Let it be true for $r < m$. Then,

$$\mathbb{P}_p\left(\left(A_1^c \cdots \cap A_{m-1}^c\right) \cap A_m^c\right)$$

$$= \mathbb{P}_p\left((A_1 \cup \cdots A_{m-1})^c \cap A_m^c\right)$$

$$= 1 - \mathbb{P}_p((A_1 \cup \cdots \cup A_{m-1})) - \mathbb{P}_p(A_m)$$

$$+ \mathbb{P}_p((A_1 \cup \cdots \cup A_{m-1}) \cap A_m)$$

Now since $\{A_i\}$ are all inc, $\bigcup_{i=1}^{m-1} A_i$ is inc

Now by FKG, $\mathbb{P}_p((A_1 \cdots \cup A_{m-1}) \cap A_m)$

$$\geq \mathbb{P}_p\Big(\bigcup_{i=1}^{m-1} A_i\Big) \mathbb{P}_p(A_m)$$

Thus, $\mathbb{P}\left(\bigcap_{i=1}^{m} A_i^c\right) \geq 1 - \mathbb{P}_p\left(\bigcup_{i=1}^{m-1} A_i\right) - \mathbb{P}_p(A_m) + \mathbb{P}_p\left(\bigcup_{i=1}^{m-1} A_i\right)\mathbb{P}(A_m)$

$$= \left(1 - \mathbb{P}_p\left(\bigcup_{i=1}^{m-1} A_i\right)\right)\left(1 - \mathbb{P}(A_m)\right)$$

$$= \mathbb{P}_p\left(\bigcap_{i=1}^{m-1} A_i^c\right)\mathbb{P}(A_m^c)$$

$$\underset{\text{(induction)}}{\geq} \prod_{i=1}^{m} \mathbb{P}_p(A_i^c)$$

This shows the claim, thus

$$\mathbb{P}_p\left(\bigcap_{i=1}^{r} A_i^c\right) \geq \left\{\min_i \mathbb{P}_p(A_i^c)\right\}^r$$

$$\left[\mathbb{P}_p\left(\bigcap_{i=1}^{r} A_i^c\right)\right]^{\frac{1}{r}} \geq 1 - \max_i\{\mathbb{P}_p(A_i)\}$$

$$\max_i\{\mathbb{P}_p(A_i)\} \geq 1 - \left(1 - \mathbb{P}_p\left(\bigcup_{i=1}^{r} A_i\right)\right)^{\frac{1}{r}}$$

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